International Journal Information System and Computer Science (IJISCS)

Full Paper
eISSN : 2598-246X
pISSN : 2598-0793

SOME RESULTS IN THE EXISTENCE, UNIQUENESS AND STABILITY PERIODIC SOLUTION OF NEW VOLTERRA INTEGRAL EQUATIONS WITH SINGULAR KERNEL

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Article history:
Received: 04 September 2020
Revised: 18 October 2020
Accepted: 03 November 2020

Keywords:
Numerical-analytic method;
periodic integral equations;
singular kernel;
Banach fixed theorem;

Abstract
The aim of this work is to study the existence, uniqueness and stability of periodic solutions of some classes for non-linear systems of new Volterra integral equations with singular kernel in two variables by using Riemann integrals. Furthermore, we investigation the existence, uniqueness and stability of the fundamental tools employed in the analysis are based on applications by depending on the numerical-analytic method for studying the periodic solutions of ordinary differential equations which were introduced by Samoilenko. The study of such nonlinear Volterra integral equations with singular kernel leads us to improve and extend Samoilenko method. Theus the non-linear integral equations with singular kernel that we have introduced in the study become more general and detailed than those introduced by Butris.

1.0 INTRODUCTION

Integral equation has been arisen in many mathematical and engineering field, so that solving this kind of problems are more efficient and useful in many research branches. Analytical solution of this kind of equation is not accessible in general form of equation and we can only get an exact solution only in special cases. But in industrial problems we have not spatial cases so that we try to solve this kind of equations numerically in general format. Many numerical schemes are employed to give an approximate solution with sufficient accuracy [3,4,6,7,8,9,10].

Integral equations of various types and kinds play an important role in many branches of mathematics. Over the past thirty years substantial progress has been made in developing innovative approximate analytical and purely [5,11,13].

An integral equation is a functional equation in which the unknown function appears under one or several integral signs; if, in addition, the equation contains a derivative of this function we call the equation an integral equations. In an integral equations of Volterra type the integrals containing the unknown function are characterized by a variable upper limit of integration. In this study we want to discuss on convergence of projection method with integral equation then present a numerical solution to this type of equation [11,12,13,15,16,17].

Samoilenko [12,13], assumes the numerical analytic method to study the periodic solutions for ordinary differential equations and their algorithm structure. This method includes uniformly sequences of periodic functions and the result is the use of the periodic solutions on a wide range which is different from the processes in industry and technology.

Consider the following Volterra integral equation with singular kernel which has the form:

$u(t,u_0) = f(t) + \int_\tau^t [\, f(s,u(s,u_0)),\ \int_{a(s)}^{b(s)} H(s,\tau\,)h(\tau,\eta,u(\eta,u_0))d\eta)]ds.$ … (VI)

Suppose that the functions $f(t,u,v,) = (f_1(t,u,v\ ), f_2(t,u,v\ ), \cdots, f_n(t,u,v\ ))$, $h(t,s,u) = (h_1(t,s,u), h_2(t,s,u), \cdots, h_n(t,s,u)), f(t) = (f_1(t), f_2(t), \cdots, f_n(t))$ are defined and continuous in the domain:

$(t,u,v\ ) \in R^1 \times D \times D_1 = (-\infty,\infty) \times D \times D_1,$ $\cdots(1)$

which are continuous functions in t,u,v and periodic in t of period T. Also $a(t)$ and $b(t)$ are continuous and periodic in t of period T, where D, D_1 is closed and bounded domains subsets of Euclidean space R^n.

Suppose that the functions $f(t,u,v\ )$, and $h(t,s,u)$ satisfies the following inequalities:

$$\left.\begin{aligned} &\|f(t,u,v,w)\| \le M\,, \\ &\|h(t,s,u)\| \le N \end{aligned}\right\} \qquad \cdots(2)$$

$\|f(t,u_1,v_1\ ) - f(t,u_1,v_1\ )\| \le L_1\|u_1 - u_2\| + L_2\|v_1 - v_2\|$; $\cdots(3)$

$\|h(t,s,u_1) - h(t,s,u_2)\| \le L_3\|u_1 - u_2\|$. $\cdots(4)$

$\forall\, t \in R^1, u, u_1, u_2 \in D, v, v_1, v_2 \in D_1$ where M, N, L_1, L_2 and L_3 are positive constants. Furthermore whose kernel function $H(t,s)$: $\mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$ is singular which is defined ,continuous and periodic in t, s and satisfy the following conditions

$$\int_{a(t)}^{b(t)} \|H(t,s)\| ds \le Q < \infty\ , \qquad \cdots(5)$$

where $-\infty < \tau \le s \le t \le \tau + T < \infty$ Q is positive constant and $\|.\| = \max_{t\in[\tau,\tau+T]\in[0,T]} |.|$.

We define a non-empty sets

$$\left.\begin{aligned} D_f &= D - \frac{T}{2}MQ\ , \\ D_{1f} &= D_1 - \frac{T}{2}ML_3\,Q\ . \end{aligned}\right\} \qquad \cdots(6)$$

Moreover, we suppose that the greatest value of the following equation $q = \frac{T}{2}\,[L_1 + L_2 + L_3\,Q\ ]$, does not exceed unity, i. e.

$q < 1\,.$ $\cdots(7)$

By using Lemma 3.1[13],we can state and proof the following:

Lemma 1. Let $f(t,u,v\ )$ be a vector function which is defined in the interval $t \in [\tau, \tau + T]$ then:

$\|L(t,u_0)\| \le \beta(t)M$ $\cdots(8)$

where $\beta(t) = 2(t-\tau)(1 - \frac{t-\tau}{T})$ and

$$L(t,u_0) = \int_\tau^t [\quad f(s,u(s,u_0)),\ \int_{a(s)}^{b(s)} H(s,\tau\,)h(\tau,\eta,u(\eta,u_0))d\eta) -$$

$$-\frac{1}{T}\int_\tau^{\tau+T} f(s,u(s,u_0)),\ \int_{a(s)}^{b(s)} H(s,\tau\,)h(\tau,\eta,u(\eta,u_0))d\eta)ds]ds$$

Proof. Assuming

$$\|L(t,u_0)\| \leq (1-\frac{t-\tau}{T})\int_{\tau}^{t}\left\|f(s,u_0(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u(\eta,u_0))d\eta\right\|ds$$
$$+\frac{t-\tau}{T}\int_{\tau}^{\tau+T}\left\|f(s,u_0(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u(\eta,u_0))d\eta\right\|ds$$

$$\leq ((1-\frac{t-\tau}{T})\int_{\tau}^{t} Mds+\frac{t-\tau}{T}\int_{\tau}^{\tau+T} Mds,$$
$$\leq \beta(t)M$$

for all $t\in[\tau,\tau+T]$ and $u_0\in D_f$.

2.0 APPROMIXMATION PERIODIC SOLUTIONS OF (VI)

The study of the approximation of periodic solution for Volterra integral equation (VI) will be introduced by the following theorem.

Theorem 1. Let $f(t,u,v), h(t,s,u)$ and $f(t)$ be vector functions which are defined, continuous and periodic of period T on the domain (1), satisfy the inequalities and condition (2) to (7), then there exist the sequence of functions:

$$u_{m+1}(t,u_0)=f(t)+\int_{\tau}^{t}[\,f(s,u_m(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u_m(\eta,u_0))d\eta)$$
$$-\frac{1}{T}\int_{\tau}^{\tau+T} f(s,u_m(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u_m(\eta,u_0))d\eta)ds]ds \qquad \cdots(9)$$

with

$u_0(t,u_0)=f(t)\qquad ,m=0,1,2,\cdots,$

periodic in t of period T, and convergent uniformly as $m\to\infty$ in the domain:

$$(t,u_0)\in[\tau,\tau+T]\times D_f \qquad \ldots\quad (10)$$

to the limit function $u^0(t,u_0)$ defined in the domain (10) which is periodic in t of period T and satisfying the system of integral equations:

$$u(t,u_0)=f(t)+\int_{\tau}^{t}[\,f(s,u(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u_m(\eta,u_0))d\eta)$$
$$-\frac{1}{T}\int_{t}^{\tau+T} f(s,u(s,u_0),\int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u_m(\eta,u_0))d\eta)ds]ds \quad \cdots(11)$$

with

$$\|u^0(t,u_0)-u_0(t.u_0)\|\leq\beta(t)M \qquad \cdots(12)$$
$$\|u^0(t,u_0)-u_m(t.u_0)\|\leq\omega^{m}(1-\omega)^{-1}\beta(t)M \qquad \cdots(13)$$

for all $m\geq 0$ and $t\in[\tau,\tau+T]$.

Proof. Consider the sequence of functions $u_1(t,u_0),\ u_2(t,u_0),\ \cdots,u_m(t,u_0),\ \cdots,$ defined by the recurrences relation (9), each of these functions are defined and continuous in the domain (1) and periodic in t of period T.

Now, by using (10) and Lemma 1, when m=0, we get

$\|u_1(t,x_0)-u_0(t,u_0)\|=$

$$\leq \left(1-\frac{t-\tau}{T}\right)\int_{\tau}^{t}\|f(s,u_0(s,u_0),\int_{a(s)}^{b(s)}H(s,\tau)h(\tau,\eta,u(\eta,u_0))d\eta\|ds$$

$$+\frac{t-\tau}{T}\int_{t}^{\tau+T}\|f(s,u_0(s,u_0),\int_{a(s)}^{b(s)}H(s,\tau)h(\tau,\eta,u(\eta,u_0))d\eta\|\ ds$$

$\leq \beta(t)M$

and hence

$$\|u_1(t,x_0)-u_0(t,u_0)\| \leq \frac{T}{2}M\,. \qquad \cdots(14)$$

From (21), we have

$$\|v_1(t,u_0)-v_0(t,u_0)\| = \|\int_{a(t)}^{b(t)}H(t,s)h(t,s,u_1(s,u_0))\,ds-\int_{a(t)}^{b(t)}H(t,s)h(t,s,u_0)ds\,\|$$

$$\leq \int_{a(t)}^{b(t)}\|H(t,s)\|\,\|u_1(s,u_0)-u_0\|ds$$

$$\leq L_3\frac{T}{2}MQ$$

and hence

$$\|v_1(t,u_0)-v_0(t,u_0)\| \leq L_3\frac{T}{2}MQ \qquad \cdots(15)$$

for all $t\in[\tau,\tau+T]$, $u_0\in D_f$ and $v_0(t,u_0)=\int_{a(t)}^{b(t)}h(t,s,u_0)\,ds\in D_{1f}$ i.e. $v_1(t,u_0)\in D_1$, when $u_0\in D_f$.

Thus by mathematical induction, we have

$$\|u_m(t,u_0)-u_0\| \leq M\,\beta(t) \leq \frac{T}{2}M \qquad \cdots(16)$$

i.e. $u_m(t,u_0)\in D$ for all $t\in[\tau,\tau+T], u_0\in D_f$.

Now from (16), gives

$$\|v_m(t,u_0)-v_0(t,u_0)\| \leq L_3\frac{T}{2}MQ \qquad \cdots(17)$$

i.e. $v_m(t,u_0)D_1$, for all $t\in[\tau,\tau+T], u_0\in D_f$.

where $v_m(t,u_0)=\int_{a(t)}^{b(t)}H(t,s)h(t,s,u_m(s,u_0))\,ds$ forall $m=0,1,2,\ldots.$

We claim that the sequence of functions $u_m(t,u_0)$ is uniformly convergent on the domain (10).

For $m = 1$ in (9) and using Lemma 1, we find that

$\|u_2(t,u_0)-u_1(t,u_0)\|$

$$\leq (1-\frac{t-\tau}{T})\int_{\tau}^{t}[L_1\|u_1(s,u_0)-u_0\| + L_2(\|u_1(s,u_0)-u_0\|)+L_3(Q\|u_1(s,u_0)-u_0\|)ds +$$

$$+\frac{t-\tau}{T}\int_{t}^{\tau+T}[L_1\|u_1(s,u_0)-u_0\|+L_2(\|u_1(s,u_0)-u_0\|)+L_3(Q\|u_1(s,u_0)-u_0\|)ds$$

$$\leq [L_1+L_2+L_3\,Q\,]M\frac{T}{2}[(1-\frac{t-\tau}{T})\int_{\tau}^{t}ds+\frac{t-\tau}{T}\int_{t}^{\tau+T}ds]\,.$$

$$\leq \beta(t)\,M\frac{T}{2}[L_1+L_2+\ L_3Q].$$

and hence

$\|u_2(t,u_0)-u_1(t,u_0)\| \leq \omega\beta(t)M.$

Suppose that the following inequality

$$\|u_{k+1}(t,u_0)-u_k(t,u_0)\| \leq\ \omega^k\beta(t)\,M \qquad \cdots(18)$$

is holds for some $m=k$, then we shall to prove that:

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \le \omega^{k+1}\beta(t)\,M\,. \qquad \cdots(19)$$

From (18) and using lemma1, when $m = k+1$ and the inequality (19) we get:

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \le (1-\frac{t}{T})\int_{\tau}^{t} [L_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\|$$
$$+L_2L_3(Q\|u_{k+1}(s,u_0)-u_k(s,u_0)\|)]ds$$
$$+\frac{t}{T}\int_{t}^{\tau+T} [L_1\|u_{k+1}(s,u_0)-u_k(s,u_0)\| +$$
$$+L_2L_3(Q\|u_{k+1}(s,u_0)-u_k(s,u_0)\|)]ds$$

$$\le \frac{T}{2}\,[L_1+L_2+L_3Q\,]\omega^k\beta(t)\,M$$
$$\le \omega^{k+1}\beta(t)\,M$$

So that

$$\|u_{k+2}(t,u_0)-u_{k+1}(t,u_0)\| \le \omega^{k+1}\beta(t)\,M$$

By mathematical induction and by (18) and (19) the following inequality is holds:

$$\|u_{m+1}(t,u_0)-u_m(t,u_0)\| \le \omega^m\beta(t)M \qquad \cdots(20)$$

where $\omega = \frac{T}{2}\,[L_1+L_2+L_3\,Q\,]$, for all $m = 0,1,2,\cdots$

From (20) we conclude that for $k \ge 0$,
we have the following inequality:

$$\|u_{m+k}(t,u_0)-u_m(t,u_0)\| \le \|u_{m+k}(t,u_0)-u_{m+k-1}(t,u_0)\|$$
$$+\|u_{m+k-1}(t,u_0)-x_{m+k-2}(t,u_0)\| + \cdots + \|u_{m+1}(t,u_0)-u_m(t,u_0)\|$$
$$\le \omega^{m+k-1}\|u_1(t,u_0)-u_0\| + \omega^{m+k-2}\|u_1(t,u_0)-u_0\| + \cdots + \omega^m\|u_1(t,u_0)-u_0\|$$
$$\le \omega^m\big(1+\omega+\omega^2+\cdots+\omega^{k-2}+\omega^{k-1}\big)\|u_1(t,u_0)-u_0\|$$

Therefore

$$\|u_{m+k}(t,u_0)-u_m(t,u_0)\| \le \omega^{m}(1-\omega)^{-1}\beta(t)M\,. \qquad \cdots(21)$$

for all $t \in [\tau,\tau+T], u_0 \in D_f$.

By using the condition (7) and the inequality (21), we find that

$$\lim_{m\to\infty}\omega^m = 0 \qquad \cdots(22)$$

The relation (22) and (23) prove the uniform convergence of the sequence of functions (9) in the domain (10) as $m \to \infty$.

Let

$$\lim_{m\to\infty} u_m\,(t,u_0) = u^0\,(t,u_0) \qquad \cdots(23)$$

Since the sequence of functions $u_m(t,u_0)$ is periodic in t of period T, Then the limiting function $u^0(t,x_0)$ is also periodic in t of period T.

Moreover, by the hypotheses and conditions of the theorem, the inequalities (12) and (13) are satisfied for all $m \ge 0$.

Theorem 2. With the hypotheses and all conditions of the theorem 1, the periodic solution of Volterra integral equation (VI) is a unique on the domain (1).

Proof. Let $u^*(t,u_0)$ be another periodic solution of Volterra integral equation (VI), i. e.

$$u^*(t,u_0) = f(t) + \int_{\tau}^{t}[\,f(s,u^*(s,u_0)),\int_{a(s)}^{b(s)} H(s,\tau)h\big(\tau,\eta,u(\eta,u_0)\big)d\eta)$$
$$-\frac{1}{T}\int_{t}^{\tau+T} f(s,u^*(s,u_0)),\int_{a(s)}^{b(s)} H(s,\tau)h\big(\tau,\eta,u(\eta,u_0)\big)d\eta)ds\,]ds$$

and then we have

$\|u(t,u_0) - u^*(t,u_0)\|$

$$\leq (1-\frac{t}{T})\int_{\tau}^{t} [L_1\|u(s,u_0) - u^*(s,u_0)\|$$

$$+\frac{t}{T}\int_{t}^{\tau+T} [L_1\|u(s,u_0) - u^*(s,u_0)\| + L_2L_3(Q\|u(s,u_0) - u^*(s,u_0)\|)ds$$

$$\leq \frac{T}{2}[L_1 + L_2 + L_3Q\ ]\|u(t,u_0) - u^*(t,u_0)\|,$$

so that

$\|u(t,u_0) - u^*(t,u_0)\| \leq \omega\|u(t,u_0) - u^*(t,u_0)\|.$

By iteration we find that

$\|u(t,u_0) - u^*(t,u_0)\| \leq \omega^m\|u(t,u_0) - u^*(t,u_0)\|$

But from the condition (15), we get $\Lambda^m \to 0$ when $m \to \infty$, hence we obtain that $u(t,u_0) = u^*(t,u_0)$. In other words $u(t,u_0)$ is a unique periodic solution of (1). ■

3.0 EXISTENCE PERIODIC SOLUTION OF (VI)

The problem of existence of periodic solution of period T of (VI) is uniquely connected with existence of zero of the function $\Delta(0,u_0) = \Delta$ which has the form:

$$\Delta(t,u_0) = \frac{1}{T}\int_{\tau}^{\tau+T} f(t,u^0(t,u_0), \int_{a(t)}^{b(t)} H(t,s)h(t,s,u^0(s,u_0))d\tau)\,dt \qquad \cdots(24)$$

where $u^0(t,u_0)$ is the limiting function of the sequence of functions $u_m(t,u_0)$.

$$\Delta_m(t,u_0) = \frac{1}{T}\int_{\tau}^{\tau+T} f(t,u_m(t,u_0), \int_{a(t)}^{b(t)} H(t,s)h(t,s,u_m(s,u_0))d\tau)\,dt \quad \cdots(25)$$

for all $m = 0,1,2,\cdots$

Theorem 3. Let all assumptions and conditions of theorem 1 and 2 are satisfied, then the following inequality is satisfied:

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \leq \omega^{m+1}(1-\omega)^{-1}M \qquad \cdots(26)$$

for all $m \geq 0$, $u_0 \in D_f$.

Proof. By the the functions (24) and (25) we get

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\|$$

$$\leq \frac{1}{T}\int_{\tau}^{\tau+T} \|f(t,u^0(t,u_0), \int_{a(t)}^{b(t)} H(t,s)h(t,s,u^0(s,u_0))ds)\,dt$$

$$- f(t,u_m(t,u_0),, \int_{a(t)}^{b(t)} H(t,s)h(t,s,u_m(s,u_0))ds)\ \|\ ds.$$

From the inequalities (3) to (8), we get:

$$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \leq [L_1 + L_2 + L_3Q\,]\frac{1}{T}\int_{\tau}^{\tau+T} \|u^0(t,u_0) - u_m(t,u_0)\|\,dt$$

$\leq \omega^{m+1}(1-\omega)^{-1}M.$

But $\omega = [L_1 + L_2Q_1 + L_3Q_2]$, thus the above inequality can be written as:

$\|\Delta(0,u_0) - \Delta_m(0,u_0)\| \leq \omega^{m+1}(1-\omega)^{-1}M$, i. e. the inequality (37) satisfied for all $m \geq 0$.

Theorem 4. Let (VI) be defined on the interval [a, b]. Suppose that for $m \geq 0$, the function $\Delta_m(0,u_0)$ defined according to formula (25) satisfies the inequalities:

$$\left.\begin{array}{c} \min\ \Delta_m(0,u_0) \le -\ \rho_m\ , \\ a+P \le u_0 \le b-P \\ \max\ \Delta_m(0,u_0) \ge \rho_m\ , \\ a+P \le u_0 \le b-P \end{array}\right\} \quad \cdots (27)$$

Then the system (1) has periodic solution $u = u(t, u_0)$ for which $u_0 \in [a + P, b - P]$, where $P = M\frac{T}{2}$ and $\rho_m = \omega^{m+1}(1-\omega)^{-1}M$

Proof: Let u_1, u_2 be any two points in the interval $[a + P, b - P]$ such that:

$$\left.\begin{array}{c} \Delta_m(0,u_1) = \min\ \Delta_m(0,u_0)\ , \\ a+P \le u_0 \le b-P \\ \Delta_m(0,u_2) = \max\ \Delta_m\ (0,u_0)\ , \\ a+P \le u_0 \le b-P \end{array}\right\} \quad \cdots (28)$$

Taking in to account inequalities (27) and (28), we have

$$\left.\begin{array}{c} \Delta(0,u_1) = \Delta_m(0,u_1) + [\Delta(0,u_1) - \Delta_m(0,u_1)] \le 0\ , \\ \Delta(0,u_2) = \Delta_m(0,u_2) + [\Delta(0,u_2) - \Delta_m(0,u_2)] \ge 0 \end{array}\right\} \quad \cdots (29)$$

It follows from the inequalities (29) and the continuity of the function $\Delta(0, u_0)$, that there exists an isolated singular point u^0 ,$u^0 \in [u_1, u_2]$, such that $\Delta(0, u^0) = 0$. This means that the Volterra integral equation (VI) has a periodic solution $u = u(t, u_0)$ for which $u_0 \in [a + P, b - P]$.

Remark 1. Theorem 4 is proved when u_0 is a scalar singular point which should be isolated (For this remark, see [12]).

4.0 STABILITY PERIODIC SOLUTION OF (VI)

In this section, we study the stability of a periodic solution for the integral equation (VI).

Theorem 5. If the function $\Delta(0, u_0)$ is defined by $\Delta: D_f \to R^n$, and by the equation (24), where $u^0(t, u_0)$ is a limit of the sequence function $\{u_m(t, u_0)\}_{m=0}^{\infty}$. Then the following inequalities hold:

$$\|\Delta(0,u_0)\| \le M \quad \cdots (30)$$

and

$$\|\Delta(0,u_0^1) - \Delta(0,u_0^2)\| \le \frac{2}{T}\omega(1-\omega)^{-1}\|f^1(t) - f^2(t)\| \quad \cdots (31)$$

for all $u^0, u_0^1, u_0^2 \in D_f$ and E is identity matrix.

Proof. From the properties of the function $u^0(t, u_0)$ as in theorem 1, the function $\Delta(t, u_0)$ is continuous and bounded by M in the domain $R^1 \times D_f$.

By using the function (24), we have

$$\|\Delta(0,u_0^1) - \Delta(0,u_0^2)\| =$$

$$\le \frac{1}{T}\int_{\tau}^{\tau+T} \|f(t,u^0(t,u_0^1), \int_{a(t)}^{b(t)} H(t,s)h(t,s,u^0(s,u_0^1))ds) \ - f(t,u^0(t,u_0^2), \int_{a(t)}^{b(t)} H(t,s)h(t,s,u^0(s,u_0^2))ds)\ \|\, dt$$

So that

$$\|\Delta(0,u_0^1) - \Delta(0,u_0^2)\| \le\ [L_1 + L_2 + L_3 Q\,]\frac{2}{T}\int_{\tau}^{\tau+T} \|u^0(t,u_0^1) - u^0(t,u_0^2)\|\, dt$$

and hence

$$\|\Delta(0,u_0^1) - \Delta(0,u_0^2)\| \le \frac{2}{T}\ \omega\|u^0(t,u_0^1) - u^0(t,u_0^2)\| \quad \cdots (32)$$

where $u^0(t, u_0^1)$, $u^0(t, u_0^1)$ are the solution of the integral equation

$$u(t,u_0^k) = f^k(t) + \int_{\tau}^{t} [f(s,u(s,u_0^k), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u(\eta,u_0^k))d\eta)$$

$$-\frac{1}{T}\int_{\tau}^{\tau+T} f(s,u(s,u_0^k), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,u(\eta,u_0^k))d\eta)ds]ds \cdots (33)$$

with
$u_0^k(t,u_0) = f^k(t) = u_0^k$, where k=1, 2.

From (33), we get:

$$\|u^0(t,u_0^1) - u^0(t,u_0^2)\|$$

$$\leq \|f^1(t) - f^2(t)\| + \left(1 - \frac{t-\tau}{T}\right)\int_{\tau}^{t} [L_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\| + (L_2 + L_3 Q)\|u^0(s,u_0^1) - u^0(s,u_0^2)\|)]ds$$

$$+\frac{t-\tau}{T}\int_{t}^{\tau+T} [L_1\|u^0(s,u_0^1) - u^0(s,u_0^2)\| + +(L_2 + L_3 Q)\|u^0(s,u_0^1) - u^0(s,u_0^2)\|)]ds$$

$$\leq \|f^1(t) - f^2(t)\| + \omega\|u^0(t,u_0^1) - u^0(t,u_0^2)\|.$$

So that:

$$\|u^0(t,u_0^1) - u^0(t,u_0^2)\| \leq (1-\omega)^{-1}\|f^1(t) - f^2(t)\| \qquad \cdots (34)$$

For all $t \in [0,T]$, $u_0^1, u_0^1 \in D_f$.

So, substituting inequality (34) in the inequality (33) we get the inequality (31).

Remark 2. Theorem 5, confirms the stability of the solution for the system (1), that is when a slight change happens in the point u_0 , then a slight change will happen in the function $\Delta(0,u_0)$. For this remark see [8].

5.0 BANACH FIXEDPOINT THEOREM

In this section we study the existence and uniqueness periodic solution of integral equation (VI) by the following:

Lemma 2.[1]. Let S be a space of all continuous function on R^1 , for any $z \in s$ define $\|z\|$ by $\|z\| = \max_{t\in[\tau,\tau+T]}|z(t)|$. Then $(s,\|z\|)$ is a Banach space.

Theorem 6.[1]. Let E be a Banach space. If T^* is a contraction mapping on E Then T^* has one and only one fixed point in E.

Theorem 7. Let $f(t,u,v,)$, $h(t,s,u)$ and $f(t)$ be vectors functions which are defined and continuous and periodic in t of period T on the domain (1) and satisfying all inequalities and conditions of the theorem 1 and 2.

Then the integral equation (VI) has a unique periodic continuous solution $z(t,u_0)$ on the domain (2), provided that $q = \frac{T}{2}[L_1 + +L_2 + L_3Q]$.

Proof. Let $(C(G),\|.\|)$ is a Banach space, where $G = \{(t,u,v);t \in R^1, u \in D, v \in D_1\}$,
Define a mapping T^* on G by

$$T^*z(t,u_0) = f(t) + \int_{\tau}^{t}[f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(t,s)h(s,\tau,z(\tau,u_0))d\tau)$$

$$-\frac{1}{T}\int_{t}^{\tau+T} f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(t,s)h(s,\tau,z(\tau,u_0))d\tau)ds]ds$$

Since $f(t)$, $f(t,z,v)$and H(t,s) are continuous on the domain (2), then $\int_{a(t)}^{b(t)} H(t,s)h(t,s,z(s,u_0))ds$ are also continuous on the domain (2). domain,

So that

$$\int_{\tau}^{t}[f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,z(s,u_0))d\eta)$$

$$-\frac{1}{T}\int_{t}^{\tau+T} f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,z(s,u_0))d\eta)ds]ds.$$

is also continuous on same domain .

Thus $T^{*}z(t,u_0)$ is continuous on the domain(1).

Hence

$T^{*}z(t,u_0): G \to G$

Next we claim that $T^{*}z(t,u_0)$ is a contraction mapping on G, let $z(t,u_0), w(t,u) \in G$, Then

$\|T^{*}z(t,u_0) - T^{*}w(t,u_0)\| \le \omega \max_{t\in[\tau,\tau+T]}\{|z(t,u_0) - w(t,u_0)|\}$

So that

$\|T^{*}z(t,u_0) - T^{*}w(t,x_0)\| \le \omega\|z(t,u_0) - w(t,x_0)\|$.

Since $0 < \omega < 1$, we find T^{*} is a contraction mapping on $t \in [\tau, \tau + T]$then by theorem 6, T^{*} has a unique fixed point $z(t,x_0) \in t \in [\tau, \tau + T]$i. e.

$T^{*}z(t,x_0) = z(t,x_0)$ and

$$z(t,u_0) = f(t) + \int_{\tau}^{t}[\, f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,z(s,u_0))d\eta)$$
$$-\frac{1}{T}\int_{t}^{\tau+T} f(s,z(s,u_0)), \int_{a(s)}^{b(s)} H(s,\tau)h(\tau,\eta,z(s,u_0))d\eta)ds]$$

Hence $z(t,u_0)$ is the unique continuous solution for the integral equation (VI) on the domain (1).

6.0 CONCLUSION

This paper provided some results in the existence, uniqueness and stability periodic solution of new Volterra integral equation with singular kernel. Theorems on existence and uniqueness and stability periodic solution are established under some necessary and sufficient conditions on closed and bounded domains (compact spaces). The numerical-analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by (Samoilenko, A. M.)

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