International Journal Information System and Computer Science (IJISCS)

Full Paper
eISSN : 2598-246X
pISSN : 2598-0793

EXISTENCE, UNIQUENESS AND STABILITY SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENT AND SYMMETRIC MATRICES

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Article history:
Received December 22, 2020
Revised Maret 4, 2021
Accepted April 5, 2021

Keywords:
Volterra integro-differential equation, retarded argument, symmetric matrices, Picard approximation method, Banach fixed point theorem.

Abstract
In this study we investigate the existence, uniqueness and stability solutions of Volterra integro-differential equations with retarded argument and symmetric matrices. The Picard approximation method and Banach fixed point theorem have been used in this study. Theorems on the existence and uniqueness of a solution are established under some necessary and sufficient conditions on closed and bounded domains (compact spaces).

INTRODUCTION

Integro-differential equations of various type and kinds play an important role in many branches of mathematics. Over the past thirty years substantial progress has been made in developing innovative approximate solutions techniques to a large class of integro-differential equation. In recent years, integro-differential equations arise in many problems of mathematical physics [1,2,9,10,11,12,13], such as the theory of elasticity, visco elasticity, or hydrodynamics. Many real-life problems that have, in the past, sometimes for differential equations actually involve a significant memory effect that can be represented in a more refined model, using a differential equation incorporating retarded or delay arguments [6,7,8 14,15,16,17,18]. The last few decades have seen an expanding interest in problems variously classified as retarded differential equations, or neutral delay differential equations. (Stochastic, whose basic numerical are addressed in [3,4,5,19,20,21,22,]). Among the application areas are the biosciences, economics, materials science, medicine, public health, and robotics, in a number of these there is an underlying problem in control theory [23,24,25,26,27,28]. In this paper, we intend to study the existence, uniqueness and stability solution for the following Volterra integro-differential equations with retarded argument and symmetric matrices:

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax + f(t) + n(t, x(t), x(t-h), y(t), y(t-h), v) \\ \frac{dy}{dt} &= By + g(t) + m(t, x(t), x(t-h), y(t), y(t-h), u) \end{aligned}\right\} \quad \dots(1)$$

where $f(t) = f_1(t) + t^{\beta} f_2(t), and g(t) = g_1(t) + t^{\beta} g_2(t)$, $\beta = 1-\alpha$,

$0 < \alpha < 1$.

Let

$$v(t) = \int_{-\infty}^{t} \frac{F(t,s)}{(t-s)^{\alpha}} \Xi\big(s, x(s), x(s-h), y(s), y(s-h)\big) ds$$

and

$$u(t) = \int_{-\infty}^{t} \frac{G(t,s)}{(t-s)^{\alpha}} \mu\big(s, x(s), x(s-h), y(s), y(s-h)\big) ds$$

Also $A = (A_{ij})$ and $B = (B_{ij})$ are non-negative matrices.

The vector functions $n(t,x,y,z,w,v)$ and $m(t,x,y,z,w,u)$ is defined and continuous on the domains:

$$\left.\begin{array}{l} (t,x,y,z,w,v) \in R^1 \times D \times D_1 \times D_2 \times D_3 \times D_v \\ (t,x,y,z,w,u) \in R^1 \times D \times D_1 \times D_2 \times D_3 \times D_u \end{array}\right\} \quad \ldots (2)$$

Where D, D_1, D_2, D_3 are closed and bounded domains subsets of Euclidean space R^n and D_v, D_u are bounded domains subset of the Euclidean space R^m.

Suppose that the vector functions $n(t,x,y,z,w,v)$ and $m(t,x,y,z,w,u)$ satisfy the following i

nequalities:

$$\begin{aligned} \|n(t,x_1,y_1,z_1,w_1,v_1) - n(t,x_2,y_2,z_2,w_2,v_2)\| \\ \le K_1\|x_1-x_2\| + K_2\|y_1-y_2\| + K_3\|z_1-z_2\| + K_4\|w_1-w_2\| \\ + K_5\|v_1-v_2\| \end{aligned} \quad .. (3)$$

$$\|m(t,x_1,y_1,z_1,w_1,u_1) - m(t,x_2,y_2,z_2,w_2,u_2)\| \le L_1\|x_1-x_2\| + L_2\|y_1-y_2\| + L_3\|z_1-z_2\| + L_4\|w_1-w_2\| + L_5\|u_1-u_2\| \quad \ldots (4)$$

$$\begin{aligned} \|\Xi(t,x_1,y_1,z_1,w_1) - \Xi(t,x_2,y_2,z_2,w_2)\| \\ \le Q_1\|x_1-x_2\| + Q_2\|y_1-y_2\| + Q_3\|z_1-z_2\| \\ + Q_4\|w_1-w_2\| \end{aligned} \quad \ldots (5)$$

$$\begin{aligned} \|\mu(t,x_1,y_1,z_1,w_1) - \mu(t,x_2,y_2,z_1,w_1)\| \\ \le J_1\|x_1-x_2\| + J_2\|y_1-y_2\| + J_3\|z_1-z_2\| \\ + J_4\|w_1-w_2\| \end{aligned} \quad \ldots (6)$$

$$\|F(t,s)\| \le \delta_1 e^{-\lambda_1(t-s)} \quad \ldots (7)$$

$$\|G(t,s)\| \le \delta_2 e^{-\lambda_2(t-s)} \quad \ldots (8)$$

$$\|f(t)\| = \|f_1(t) + t^{\beta} f_2(t)\| \le \|f_1(t)\| + \|t\|^{\beta}\|f_2(t)\| \le M_1 + T^{\beta} M_2 \quad \ldots (9)$$

$$\|g(t)\| = \|g_1(t) + t^{\beta} g_2(t)\| \le \|g_1(t)\| + \|t\|^{\beta}\|g_2(t)\| \le N_1 + T^{\beta} N_2 \quad \ldots (10)$$

where

$\|f_1(t)\| \le M_1$, $\|f_2(t)\| \le M_2$, $\|g_1(t)\| \le N_1$, $\|g_2(t)\| \le N_2$

Also

$$\|n(t,x,y,z,w,v)\| \le M, \ \|m(t,x,y,z,w,u)\| \le N \quad \ldots (11)$$

$$\|\Xi(t,x,y,z,w)\| \le M_3, \ \|\mu(t,x,y,z,w)\| \le N_3 \quad \ldots (12)$$

For all $t \in R^1, x, x_1, x_2 \in D, y, y_1, y_2 \in D_1, z, z_1, z_2 \in D_2, w, w_1, w_2 \in D_3$ and $v, v_1, v_2 \in D_v$ and $u, u_1, u_2 \in D_u$

where $M, M_1, M_2, M_3, N, N_1, N_2, N_3, K_1, K_2, K_3, K_4, K_5, L_1, L_2, L_3, L_4, L_5,$

$Q_1, Q_2, Q_3, Q_4, J_1, J_2, J_3$ and J_4 are positive constants.

and

$$\left\|e^{A(t-s)}\right\| \le \delta_3 \qquad \dots (13)$$

$$\left\|e^{B(t-s)}\right\| \le \delta_4 \qquad \dots (14)$$

where $\delta_1, \delta_2, \delta_3$ and δ_4 are positive constants, $\|.\| = \max_t |.|$.

We define the non-empty sets as follows:-

$$\left.\begin{array}{l} D_n = D - \delta_3 T\left[\left(M_1 + T^\beta M_2\right) + M\right] \\ D_n^* = D_1 - \delta_3 (T-h)\left[\left(M_1 + T^\beta M_2\right) + M\right] \\ D_m = D_2 - \delta_4 T\left[\left(N_1 + T^\beta N_2\right) + N\right] \\ D_m^* = D_3 - \delta_4 (T-h)\left[\left(N_1 + T^\beta N_2\right) + N\right] \end{array}\right\} \qquad \dots (15)$$

Furthermore, we suppose that the largest eigen-value of the matrix

$$\Lambda = \begin{pmatrix} TE_1 & TE_2 & TE_3 & TE_4 \\ T\varphi_1 & T\varphi_2 & T\varphi_3 & T\varphi_4 \\ (T-h)E_1 & (T-h)E_2 & (T-h)E_3 & (T-h)E_4 \\ (T-h)\varphi_1 & (T-h)\varphi_2 & (T-h)\varphi_3 & (T-h)\varphi_4 \end{pmatrix}$$ does not exceed unity

$$\lambda_{max}(\Lambda) = \frac{\Psi_1 + \sqrt{{\Psi_1}^2 - 4\Psi_2}}{2} < 1 \qquad \dots (16)$$

where $\Psi_1 = TE_1 + T\varphi_2 + (T-h)E_3 + (T-h)\varphi_4$

$$\Psi_2 = TE_1\,(T-h)\varphi_4 + T^2 E_1 \varphi_2 + T\varphi_2 (T-h) E_3 + (T-h)^2 E_3 \varphi_4 + (T-h)^2 \varphi_3\, E_4 + (T-h) E_2 T \varphi_3 - T\varphi_1 (T-h) E_2 - (T-h)\varphi_1 T E_4$$

$$E_1 = \delta_3 K_1 + \delta_3 K_5 Q_1 \frac{\delta_1}{2\lambda_1 T^\alpha}, E_4 = \delta_3 K_4 + \delta_3 K_5 Q_4 \frac{\delta_1}{2\lambda_1 T^\alpha}$$

$$\varphi_1 = \delta_4 L_1 + \delta_4 L_5 J_1 \frac{\delta_2}{2\lambda_2 T^\alpha}, \varphi_2 = \delta_4 L_2 + \delta_4 L_5 J_2 \frac{\delta_2}{2\lambda_2 T^\alpha}$$

$$\varphi_3 = \delta_4 L_3 + \delta_4 L_5 J_3 \frac{\delta_2}{2\lambda_2 T^\alpha}, \varphi_4 = \delta_4 L_4 + \delta_4 L_5 J_4 \frac{\delta_2}{2\lambda_2 T^\alpha}$$

We define the sequence of functions $\{x_i(t), y_i(t)\}_{i=0}^{\infty}$ on the domains (2) by the following:-

$$x_{i+1}(t) = x_0 + \int_0^t e^{A(t-s)}\Big[f(s) + n\Big(s, x_i(s), x_i(s-h), y_i(s), y_i(s-h), \int_{-\infty}^{s} \frac{F(s,\tau)}{(s-\tau)^\alpha} \Xi\big(\tau, x_i(\tau), x_i(\tau-h), y_i(\tau), y_i(\tau-h)\big)d\tau\Big)ds\Big] \qquad \dots (17)$$

with

$x_0(t) = x_0, \quad i = 0,1,2,\dots$

$$y_{i+1}(t) = y_0 + \int_0^t e^{B(t-s)}\Big[g(s) + m\Big(s, x_i(s), x_i(s-h), y_i(s), y_i(s-h), \int_{-\infty}^{s} \frac{G(s,\tau)}{(s-\tau)^\alpha} \mu\big(\tau, x_i(\tau), x_i(\tau-h), y_i(\tau), y_i(\tau-h)\big)d\tau\Big)ds\Big] \qquad \dots (18)$$

with

$y_0(t) = y_0,\ i = 0,1,2,\dots$

2.0 EXISTENCE SOLUTION OF (1).

In this section, we prove the existence theorem of Volterra integro-differential equation (1) by using Picard approximation method.

Theorem1. Let the vector functions $n(t,x,y,z,w,v)$and $m(t,x,y,z,w,u)$are defined continuous on the domain (2) satisfy the inequalities (3) to (14) and condition (15). Then there exist sequences of functions (17) and (18) convergent uniformly on the domain:

$$\left.\begin{matrix}(t,x_0) \in R^1 \times D_n \\ (t,y_0) \in R^1 \times D_m\end{matrix}\right\} \quad \dots (19)$$

to the limit vector function $\begin{pmatrix}x(t)\\y(t)\end{pmatrix}$which satisfy the following integral equations:

$$x(t) = x_0 + \int_0^t e^{A(t-s)}[f(s) + n(s, x(s), x(s-h), y(s), y(s-h), \int_{-\infty}^{s} \frac{F(s,\tau)}{(s-\tau)^\alpha} \Xi\big(\tau, x(\tau), x(\tau-h), y(\tau), y(\tau-h)\big)d\tau)ds] \quad \dots (20)$$

$$y(t) = y_0 + \int_0^t e^{B(t-s)}[g(s) + m(s, x(s), x(s-h), y(s), y(s-h), \int_{-\infty}^{s} \frac{G(s,\tau)}{(s-\tau)^\alpha} \mu\big(\tau, x(\tau), x(\tau-h), y(\tau), y(\tau-h)\big)d\tau)ds] \quad \dots (21)$$

which is a solution of(1), provided that

$$\left.\begin{matrix}\|x_i(t) - x_0\| \le \delta_3 T\big[\big(M_1 + T^\beta M_2\big) + M\big] \\ \|y_i(t) - y_0\| \le \delta_4 T\big[\big(N_1 + T^\beta N_2\big) + N\big]\end{matrix}\right\} \quad \dots (22)$$

and

$$\begin{pmatrix}\|x(t) - x_i(t)\| \\ \|y(t) - y_i(t)\|\end{pmatrix} \le \Lambda^i (E - \Lambda)^{-1} \Omega_1 \quad \dots (23)$$

Proof. By mathematical induction, we can prove that:

$$\left.\begin{matrix}\|x_i(t) - x_0\| \le \delta_3 T\big[\big(M_1 + T^\beta M_2\big) + M\big] \\ \|y_i(t) - y_0\| \le \delta_4 T\big[\big(N_1 + T^\beta N_2\big) + N\big]\end{matrix}\right\} \quad \dots (24)$$

Therefore $x_i(t) \in D, y_i(t) \in D_1, t \in [0,T], x_0 \in D_n, y_0 \in D_m, i = 1,2,\dots$

And also by mathematical induction, we get

$$\left.\begin{aligned}\|x_i(t-h)-x_0\| &\le \delta_3(T-h)[[(M_1+T^{\beta}M_2)+M]] \\ \|y_i(t-h)-y_0\| &\le \delta_4(T-h)[[(N_1+T^{\beta}N_2)+N]]\end{aligned}\right\} \quad \dots(25)$$

Therefore $x_i(t-h)\in D_1, y_i(t-h)\in D_3, t\in[0,T], x_0\in D_n^*,\ y_0\in D_m^*, i=1,2,\dots$

Next, we shall prove that sequences of functions (17)and (18)convergent uniformly on the domain(2).

By mathematical induction, we can prove that:

$$\begin{aligned}&\|x_{i+1}(t)-x_i(t)\| \\ &\quad \le \mathrm{T}E_1\|x_i(t)-x_{i-1}(t)\|+TE_2\|x_i(t-h)-x_{i-1}(t-h)\| \\ &\quad + TE_3\|y_i(t)-y_{i-1}(t)\| \\ &\quad + TE_4\|y_i(t-h) \\ &\quad - y_{i-1}(t-h)\| \end{aligned} \quad \dots (26)$$

and

$$\begin{aligned}&\|y_{i+1}(t)-y_i(t)\| \\ &\quad \le T\varphi_1\|x_i(t)-x_{i-1}(t)\|+T\varphi_2\|x_i(t-h)-x_{i-1}(t-h)\| \\ &\quad + T\varphi_3\|y_i(t)-y_{i-1}(t)\| \\ &\quad + T\varphi_4\|y_i(t-h)-y_{i-1}(t-h)\| \end{aligned} \quad \dots (27)$$

Also

$$\begin{aligned}&\|x_{i+1}(t-h)-x_i(t-h)\| \\ &\quad \le (\mathrm{T}-\mathrm{h})E_1\|x_i(t)-x_{i-1}(t)\|+(T-h)E_2\|x_i(t-h)-x_{i-1}(t-h)\| \\ &\quad + (T-h)E_3\|y_i(t)-y_{i-1}(t)\| \\ &\quad + (T-h)E_4\|y_i(t-h)-y_{i-1}(t-h)\| \end{aligned} \quad \dots(28)$$

and

$$\begin{aligned}&\|y_{i+1}(t-h)-y_i(t-h)\| \\ &\quad \le (T-h)\varphi_1\|x_i(t)-x_{i-1}(t)\|+(T-h)\varphi_2\|x_i(t-h)-x_{i-1}(t-h)\| \\ &\quad + (T-h)\varphi_3\|y_i(t)-y_{i-1}(t)\| \\ &\quad + (T-h)\varphi_4\|y_i(t-h)-y_{i-1}(t-h)\| \end{aligned} \quad \dots(29)$$

Rewriting inequalities (26), (27), (28) and (29) by vector form, we get

$$\Omega_{i+1}(t)\le \Lambda(t)\Omega_i \quad \dots(30)$$

$$\Omega_{i+1}=\begin{pmatrix}\|x_{i+1}(t)-x_i(t)\| \\ \|y_{i+1}(t)-y_i(t)\| \\ \|x_{i+1}(t-h)-x_i(t-h)\| \\ \|y_{i+1}(t-h)-y_i(t-h)\|\end{pmatrix}$$

$$\Omega_i=\begin{pmatrix}\|x_i(t)-x_{i-1}(t)\| \\ \|y_i(t)-y_{i-1}(t)\| \\ \|x_i(t-h)-x_{i-1}(t-h)\| \\ \|y_i(t-h)-y_{i-1}(t-h)\|\end{pmatrix}$$

and

$$\Lambda(t)=\begin{pmatrix} tE_1 & tE_2 & tE_3 & tE_4 \\ t\varphi_1 & t\varphi_2 & t\varphi_3 & t\varphi_4 \\ (t-h)E_1 & (t-h)E_2 & (t-h)E_3 & (t-h)E_4 \\ (t-h)\varphi_1 & (t-h)\varphi_2 & (t-h)\varphi_3 & (t-h)\varphi_4 \end{pmatrix}$$

Now, we take the maximal value for the both sides of the inequalities (30) we get

$$\Omega_{i+1} \le \Lambda\Omega_i \quad \dots (31)$$

where $\Lambda = \max_{t\in[0,T]} \Lambda(t)$.

By repetition(35), we find that

$$\Omega_{i+1} \le \Lambda^i\Omega_1 \quad \dots (32)$$

where $\Omega_1 = \begin{pmatrix} \delta_3 T\left[\left(M_1 + T^\beta M_2\right) + M\right] \\ \delta_4 T\left[\left(N_1 + T^\beta N_2\right) + N\right] \end{pmatrix}$

Hence

$$\sum_{l=1}^{i} \Omega_l \le \sum_{l=1}^{i} \Lambda^{l-1}\Omega_1 \quad \dots (33)$$

By using (16),then the sequence $(33)$is uniformly convergent that is

$$\lim_{i\to\infty} \sum_{l=1}^{i} \Lambda^{l-1}\Omega_1 = \sum_{l=1}^{\infty} \Lambda^{l-1}\Omega_1 = (E-\Lambda)^{-1}\Omega_1 \quad \dots (34)$$

Let

$$\lim_{i\to\infty} \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \quad \dots (35)$$

Since the sequence of functions (17) and (18) are define and continuous in the domain (2) then the limiting vector function

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is also defined and continuous on the same domain, hence the vector function $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ is a solution of (1).

UNIQUENESS SOLUTION OF (1).

In this section, we prove the uniqueness theorem of Volterra integro-differential equation (1) by using the same method in section(ii).

Theorem 2. With the hypotheses and all conditions and inequalities of the theorem 1, then the solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$is a unique on the domain (2).

Proof. Let $\begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix}$be another solution of (1).

where

$$x^*(t) = x_0 + \int_0^t e^{A(t-s)}[f(s) + n(s, x^*(s), x^*(s-h), y^*(s), y^*(s-h), \int_{-\infty}^s \frac{F(s-\tau)}{(s-\tau)^\alpha} \mho(\tau, x^*(\tau), x^*(\tau-h), y^*(\tau), y^*(\tau-h))d\tau)ds] \quad \dots (36)$$

and

$$y^*(t) = y_0 + \int_0^t e^{B(t-s)}[g(s) + m(s, x^*(s), x^*(s-h), y^*(s), y^*(s-h), \int_{-\infty}^s \frac{G(s-\tau)}{(s-\tau)^\alpha} \mu(\tau, x^*(\tau), x^*(\tau-h), y^*(\tau), y^*(\tau-h))d\tau)ds] \quad \dots. (37)$$

Taking

$$\|x(t) - x^*(t)\| \le \int_0^t \|e^{A(t-s)}\| [K_1\|x(s) - x^*(s)\| + K_2\|x(s-h) - x^*(s-h)\| + K_3\|y(s) - y^*(s)\| + K_4\|y(s-h) - y^*(s-h)\| + K_5 \frac{\delta_1}{2\lambda_1 T^\alpha}[Q_1\|x(s) - x^*(s)\| + Q_2\|x(s-h) - x^*(s-h)\| + Q_3\|y(s) - y^*(s)\| + Q_4\|y(s-h) - y^*(s-h)\|]]ds$$

Therefore

$$\|x(t) - x^*(t)\| \le TE_1\|x(t) - x^*(t)\| + TE_2\|x(t-h) - x^*(t-h)\| + TE_3\|y(t) - y^*(t)\| + TE_4\|y(t-h) - y^*(t-h)\| \quad \dots (38)$$

Now similarly

$$\|y(t) - y^*(t)\| \le T\varphi_1\|x(t) - x^*(t)\| + T\varphi_2\|x(t-h) - x^*(t-h)\| + T\varphi_3\|y(t) - y^*(t)\| + T\varphi_4\|y(t-h) - y^*(t-h)\| \quad \dots (39)$$

Also

$$\|x(t-h)-x^*(t-h)\|$$
$$\leq (T-h)E_1\|x(t)-x^*(t)\| + (T-h)E_2\|x(t-h)-x^*(t-h)\| + (T-h)E_3\|y(t)-y^*(t)\| + (T-h)E_4\|y(t-h)-y^*(t-h)\| \quad \dots (40)$$

And

$$\|y(t-h)-y^*(t-h)\|$$
$$\leq \int_0^{t-h} \left\|e^{B(t-s)}\right\| [L_1\|x(s)-x^*(s)\| + L_2\|x(s-h)-x^*(s-h)\| + L_3\|y(s)-y^*(s)\|$$
$$+ L_4\|y(s-h)-y^*(s-h)\| + L_5\frac{\delta_2}{2\lambda_2 T^\alpha}[J_1\|x(s)-x^*(s)\| + J_2\|x(s-h)-x^*(s-h)\|$$
$$+ J_3\|y(s)-y^*(s)\| + J_4\|y(s-h)-y^*(s-h)\|]]ds$$

$$\|y(t-h)-y^*(t-h)\|$$
$$\leq (T-h)\varphi_1\|x(t)-x^*(t)\| + (T-h)\varphi_2\|x(t-h)-x^*(t-h)\| + (T-h)\varphi_3\|y(t)-y^*(t)\| + (T-h)\varphi_4\|y(t-h)-y^*(t-h)\| \quad \dots (41)$$

Then we can rewrite the inequalities (38), (39), (40) and (41) by the vector form: -

$$\begin{pmatrix} \|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\| \\ \|x(t-h)-x^*(t-h)\| \\ \|y(t-h)-y^*(t-h)\| \end{pmatrix} \leq \Lambda \begin{pmatrix} \|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\| \\ \|x(t-h)-x^*(t-h)\| \\ \|y(t-h)-y^*(t-h)\| \end{pmatrix} \quad \dots (42)$$

Then by the condition (16), we find that

$$\begin{pmatrix} \|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\| \\ \|x(t-h)-x^*(t-h)\| \\ \|y(t-h)-y^*(t-h)\| \end{pmatrix} < \begin{pmatrix} \|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\| \\ \|x(t-h)-x^*(t-h)\| \\ \|y(t-h)-y^*(t-h)\| \end{pmatrix}$$

This is contradiction, then

$$\begin{pmatrix} \|x(t)-x^*(t)\| \\ \|y(t)-y^*(t)\| \\ \|x(t-h)-x^*(t-h)\| \\ \|y(t-h)-y^*(t-h)\| \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} x(t) \\ y(t) \\ x(t-h) \\ y(t-h) \end{pmatrix} = \begin{pmatrix} x^*(t) \\ y^*(t) \\ x^*(t-h) \\ y^*(t-h) \end{pmatrix}$$

And hence the solutions $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$is a unique of (1).

4.0 STABILITY SOLUTION OF(1).

In this section, we study the stability solution of the problem(1) by the following theorem:

Theorem 3. if the inequalities (3) to (14) are satisfied and $\begin{pmatrix}\bar{x}(t)\\ \bar{y}(t)\end{pmatrix}$which are another solutions of (1) then the solutions is stable for all $t \geq 0$.

where

$$\bar{x}(t) = \bar{x}_0 + \int_0^t e^{A(t-s)}[f(s) + n(s, \bar{x}(s), \bar{x}(s-h), \bar{y}(s), \bar{y}(s-h), \int_{-\infty}^{s} \frac{F(s-\tau)}{(s-\tau)^\alpha} \Xi(\tau, \bar{x}(\tau), \bar{x}(\tau-h), \bar{y}(\tau), \bar{y}(\tau-h)) d\tau) ds] \quad \text{... (43)}$$

and

$$\bar{y}(t) = \bar{y}_0 + \int_0^t e^{B(t-s)}[g(s) + m(s, \bar{x}(s), \bar{x}(s-h), \bar{y}(s), \bar{y}(s-h), \int_{-\infty}^{s} \frac{G(s-\tau)}{(s-\tau)^\alpha} \mu(\tau, \bar{x}(\tau), \bar{x}(\tau-h), \bar{y}(\tau), \bar{y}(\tau-h)) d\tau) ds] \quad \text{... (44)}$$

Proof.

$$\|x(t) - \bar{x}(t)\| = \Bigg\| x_0 + \int_0^t e^{A(t-s)}[f(s) + n(s, x(s), x(s-h), y(s), y(s-h), \int_{-\infty}^{s} \frac{F(s-\tau)}{(s-\tau)^\alpha} \Xi(\tau, x(\tau), x(\tau-h), y(\tau), y(\tau-h)) d\tau) ds] - \bar{x}_0 - \int_0^t e^{A(t-s)}[f(s) - n(s, \bar{x}(s), \bar{x}(s-h), \bar{y}(s), \bar{y}(s-h), \int_{-\infty}^{s} \frac{F(s-\tau)}{(s-\tau)^\alpha} \Xi(\tau, \bar{x}(\tau), \bar{x}(\tau-h), \bar{y}(\tau), \bar{y}(\tau-h)) d\tau) ds] \Bigg\|$$

$$\leq \|x_0 - \bar{x}_0\| + \int_0^t \|e^{A(t-s)}\| [K_1\|x(s) - \bar{x}(s)\| + K_2\|x(s-h) - \bar{x}(s-h)\| + K_3\|y(s) - \bar{y}(s)\| + K_4\|y(s-h) - \bar{y}(s-h)\| + K_5 \frac{\delta_1}{2\lambda_1 T^\alpha} [Q_1\|x(s) - \bar{x}(s)\| + Q_2\|x(s-h) - \bar{x}(s-h)\| + Q_3\|y(s) - \bar{y}(s)\| + Q_4\|y(s-h) - \bar{y}(s-h)\|]] ds$$

Therefore

$$\|x(t) - \bar{x}(t)\| \leq \|x_0 - \bar{x}_0\| + T[E_1\|x(t) - \bar{x}(t)\| + E_2\|x(t-h) - \bar{x}(t-h)\| + E_3\|y(t) - \bar{y}(t)\| + E_4\|y(t-h) - \bar{y}(t-h)\|]$$

and according to the definition of stability[22] for $\|x_0 - \bar{x}_0\| \leq \delta_1$ we get

CONCLUSION

This paper provided the existence, uniqueness, and stability solution for non-linear system of Volterra integro-differential equations with retarded argument and symmetric matrices. Picard approximation (Successive approximation) method and Banach fixed point theorem have been used in this study which were introduced by [6]. Theorems on the existence and uniqueness of a solution are established under some necessary and sufficient conditions on closed and bounded domains (compact spaces).

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