



SOME NEW RESULTS OF INITIAL BOUNDARY PROBLEM CONTAIN ABC-FRACTIONAL DIFFERENTIAL EQUATIONS OF ORDER $\alpha \in (2, 3)$

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Abstract

The purpose of this research is to investigate the existence and uniqueness of solutions for a new class of Atangana-Baleanu fractional differential equations of order $\alpha \in (2,3)$ with periodic boundary conditions. Our results are based on the fixed points of Schauder, and Banach. In addition, investigate the stability of the solution using the Hyers-Ulam stable. Finally, presented an example to satisfy all theorems studies.

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1.0 INTRODUCTION

Fractional order Across many different branches of science and engineering, differential equations have recently proven to be useful tools for modeling a wide range of phenomena. The domains of control, porous media, electromagnetism, etc. all have applications (see [1]–[4]). The most popular fractional derivatives are Caputo and Riemann-Liouville derivatives, while various scholars have defined many more types of fractional operators. A new derivative was recently defined by Caputo-Fabrizio in [5]. This issue is diminished by Atangana and Baleanu [6]. a derivative based on the generalized Mittag Leffler (ML) function The fractional derivative of the Atangana-Baleanu (AB-derivative) is known as this. This derivative describes several types of dissipative events because of the nonsingular and nonlocal behavior of the AB-derivative kernel. Various models of dissipative events are described by this derivative. The nonlocality and nonsingularity of the kernel offer a more comprehensive solution to the memory under development at various scales. The development of FDEs connected to ABC-derivatives was hugely affected by academics. their application to the mathematical modeling of issues involving dynamical systems, heat flux, fluid fluxes, and electrical circuits. [7], [8]. Theoretical modeling of crystalline materials has frequently used periodic boundary conditions (PBCs). Condensed matter computations may easily implement boundary conditions using PBCs. Unified numerical techniques that consider both periodic and periodic systems are possible [9].

In this paper, The ABC-fractional differential equations with periodic boundary conditions that are being considered as follows:

$$\begin{aligned} &({}^{ABC}D_0^\alpha x)(t) + u(t)x(t) = f(t, x(t)), \\ &2 < \alpha < 3, t \in [0, T], T > 0 \end{aligned} \tag{1.1}$$

with

$$x(0) = x(T), x'(0) = x'(T), x''(0) = x''(T). \tag{1.2}$$

On the other hand, the theory of fractional differential equations places a premium on the existence of a solution and uniqueness. Additionally, the solution's stability is just as crucial as its existence because an unstable solution is ineffective and might not deliver the necessary data for the stated area. The differential equation stability problem was put up and studied by Ulam and Hyers [10]–[12]. Between 1978 and 1988, Rassias demonstrated the Ulam-Hyers stability of

both linear and nonlinear mappings [13]. Ulam-Hyers and Ulam-Hyers-Rassias stability for fractional differential equations are discussed in [14]–[17].

There are four sections in this paper. AB fractional calculus definitions and theorems are presented in Section 2. Two subsections make up Section 3. In the first, investigate the existence of solutions using Schauder's fixed point for the problems (1.1)–(1.2), and then use the Banach fixed point theorem to show that the solution is singular. In section 4, use Hyers-Ulam stable to assess the stability of the solution. To further our objectives, we employ examples.

2.0 METHODOLOGY

Mathematical Tools

In this section, we will go over several preliminaries that will be important in the next sections.

Definition 2.1 [6]. Let $\lambda \in [0,1]$, $\vartheta' \in H'(a,b)$, where $a \leq b$, then the Caputo AB-derivative is

$$({}^{ABC}D^\lambda \vartheta)(t) = \frac{B(\lambda)}{1-\lambda} \int_0^t \vartheta'(s) E_\lambda \left[-\lambda \frac{(t-s)^\lambda}{1-\lambda} \right] ds.$$

Where E_λ is the Mittag-Leffler function, $B(\lambda)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$.

The associated fractional integral of the Caputo AB-derivative is defined by

$$({}^{AB}I^\lambda \vartheta)(t) = \frac{1-\lambda}{B(\lambda)} \vartheta(t) + \frac{\lambda}{B(\lambda)} ({}_0I^\lambda \vartheta)(t).$$

Where ${}_0I^\lambda$ is the left Riemann-Liouville fractional integral.

Theorem 2.2:[18] for $x(t)$ defined on $[a,b]$ and $a \in (m, m+1]$, for some $m \in \mathbb{N}$, we have

1. $({}^{ABR}D_a^\alpha {}^{AB}I_a^\alpha)(x(t)) = x(t)$
2. $({}^{AB}I_a^\alpha {}^{ABR}D_a^\alpha)(x(t)) = x(t) - \sum_{n=0}^{m-1} \frac{x^n(a)}{n!} (t-a)^n$
3. $({}^{AB}I_a^\alpha {}^{ABC}D_a^\alpha)(x(t)) = x(t) - \sum_{n=0}^m \frac{x^n(a)}{n!} (t-a)^n$

Theorem 2.3: [19] **Arzela Fixed Point Theorem.** Let ω be a compact Hausdorff metric space. Then $\zeta \subset M(\omega)$ is said to be relatively compact whenever ζ is equicontinuous and bounded uniformly.

3.0 RESULT

Main Result

In this section, we use Krasnoselskii's and Banach's fixed point theorems to establish the existence and uniqueness of the problem (1.1) – (1.2). First, we prove the following lemmas, which are critical for obtaining existence findings.

Lemma 3.1: Let $2 < \alpha < 3$ and $x \in C([0, T], \mathbb{R})$ we have.

1. $({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-1}x)(t) = ({}^{AB}I_0^1x)(t) - \frac{t^2}{2}x'(0) - tx(0).$
2. $({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-2}x)(t) = ({}^{AB}I_0^2x)(t) - \frac{t^2}{2}x(0).$

Where $({}^{AB}I_0^1x)(t) := \int_0^t x(s)ds$, $({}^{AB}I_0^2x)(t) := \int_0^t (t-s)x(s)ds$.

Proof (1). By using **Theorem 2.2.**, we get

$$\begin{aligned} ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-1}x)(t) &= {}^{AB}I^1 ({}^{AB}I_0^{\alpha-1} {}^{ABC}D_0^{\alpha-1}x)(t). \\ ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-1}x)(t) &= {}^{AB}I^1 \left(x(t) - \sum_{k=0}^1 \frac{x^k(0)}{k!} (t-0)^k \right). \\ ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-1}x)(t) &= ({}^{AB}I^1x)(t) - {}^{AB}I^1(x(0) + tx'(0)). \\ ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-1}x)(t) &= ({}^{AB}I_0^1x)(t) - tx(0) - \frac{t^2}{2}x'(0). \end{aligned}$$

Proof (2): By using **Theorem 2.2.**, we have

$$\begin{aligned} ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-2}x)(t) &= {}^{AB}I^2 ({}^{AB}I_0^{\alpha-2} {}^{ABC}D_0^{\alpha-2}x)(t). \\ ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-2}x)(t) &= {}^{AB}I^2 \left(x(t) - \sum_{k=0}^1 \frac{x^k(0)}{k!} (t-0)^k \right). \\ ({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-2}x)(t) &= ({}^{AB}I^2x)(t) - {}^{AB}I^2(x(0)). \end{aligned}$$

$$({}^{AB}I_0^\alpha {}^{ABC}D_0^{\alpha-2}x)(t) = ({}^{AB}I_0^2x)(t) - \frac{t^2}{2}x(0).$$

Lemma 3.2: for $x(t)$ defined on $[a, b]$, a function $x(t)$ is a solution of

$$({}^{ABC}D_a^\alpha x)(t) = f(t), \quad m < \alpha \leq m + 1 \quad (3.1)$$

$$\text{with } x(a) = c_1, x'(a) = c_2, \dots, x^{(m)}(a) = c_{m+1}, \quad t \in [a, b] \quad (3.2)$$

if and only if $x(t)$ is the solution of the following integral equation

$$x(t) = \sum_{k=0}^m \frac{x^{(k)}(a)}{k!} (t-a)^k + \frac{(m+1)-\alpha}{\beta(\alpha-m)} \left(\frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} f(s) ds \right) + \frac{\alpha-m}{\beta(\alpha-m)\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.$$

Proof. As a starting point, we provided associate integrals for both side of the equation (3.1)

$$({}^{AB}I_a^\alpha {}^{ABC}D_a^\alpha)(x(t)) = {}^{AB}I_a^\alpha(f(t)) \quad \text{using Theorem 2.2. for the left side, we get.}$$

$$x(t) - \sum_{k=0}^m \frac{x^{(k)}(a)}{k!} (t-a)^k = {}^{AB}I_a^\alpha(f(t)).$$

$$x(t) - \sum_{k=0}^m \frac{x^{(k)}(a)}{k!} (t-a)^k = I^m({}^{AB}I_a^{\alpha-m}f(t)).$$

Now we use the associated fractional integral of the Caputo AB-derivative for right side, we have.

$$x(t) - \sum_{k=0}^m \frac{x^{(k)}(a)}{k!} (t-a)^k = I^m \left(\frac{1-(\alpha-m)}{\beta(\alpha-m)} f(t) + \frac{\alpha-m}{\beta(\alpha-m)} I^{\alpha-m} f(t) \right).$$

$$x(t) = \sum_{k=0}^m \frac{x^{(k)}(a)}{k!} (t-a)^k + \frac{(m+1)-\alpha}{\beta(\alpha-m)} \left(\frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} f(s) ds \right) + \frac{\alpha-m}{\beta(\alpha-m)\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds.$$

Theorem 3.3: Let $u, f: [0, T] \rightarrow R$ be a continuous function and a function $x(t)$ is a solution of the following ABC-fractional.

$({}^{ABC}D_0^\alpha x)(t) + u(t)x(t) = f(t, x(t)), \quad 2 < \alpha < 3, \quad t \in [0, T], \quad T > 0$ and A and C are non-zero positive numbers with

$$x(0) = x(T), x'(0) = x'(T), x''(0) = x''(T).$$

if and only if $x(t)$ is the solution of the following integral equation:

$$x(t) = \int_0^T G(t, s) f(s, x(s)) ds + \int_0^T \mu(t, s) x(s) ds + \frac{1}{u(T)} f(T, x(T)), \quad u(t) \neq 0$$

Where $G(t, s)$ and $\mu(t, s)$ are green functions.

$$G(t, s) := \begin{cases} -\left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} (T-s) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (T-s)^{\alpha-1} \right) \\ -\left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} (T-s)^{\alpha-2} \right) \\ + \left(\frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \right) (T-s)^{\alpha-3}, \quad \text{if } t \leq s \leq T \\ -\left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} (T-s) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (T-s)^{\alpha-1} \right) \\ -\left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} (T-s)^{\alpha-2} \right) \\ + \left(\frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \right) (T-s)^{\alpha-3} + \\ \frac{(3-\alpha)}{\beta(\alpha-2)} (t-s) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (t-s)^{\alpha-1}, \quad \text{if } 0 \leq s \leq t \end{cases}$$

also,

$$\mu(t, s) := \begin{cases} -\left(\frac{t}{T}\right) \left(-\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (T-s)^{\alpha-1} u(s) - \frac{(3-\alpha)}{\beta(\alpha-2)} (T-s) u(s) \right) - \\ \left(\frac{t(t-T)}{2T}\right) \left(-\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} (T-s)^{\alpha-2} u(s) - \frac{(3-\alpha)}{\beta(\alpha-2)} u(s) \right) + \left(\frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)}\right) (T-s)^{\alpha-3} u(s), \\ \text{if } t \leq s \leq T \\ -\left(\frac{t}{T}\right) \left(-\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (T-s)^{\alpha-1} u(s) - \frac{(3-\alpha)}{\beta(\alpha-2)} (T-s) u(s) \right) - \\ \left(\frac{t(t-T)}{2T}\right) \left(-\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} (T-s)^{\alpha-2} u(s) - \frac{(3-\alpha)}{\beta(\alpha-2)} u(s) \right) + \left(\frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)}\right) (T-s)^{\alpha-3} u(s) \\ -\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} (t-s)^{\alpha-1} u(s) - \frac{(3-\alpha)}{\beta(\alpha-2)} (t-s) u(s), \quad \text{if } 0 \leq s \leq t \end{cases}$$

Proof. We have

$$({}^{ABC}D_0^\alpha x)(t) + u(t) x(t) = f(t, x(t))$$

By using **Lemma 3.1** and **Lemma 3.2**, we get

$$x(t) = x(0) + tx'(0) + \frac{t^2}{2} x''(0) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s) f(s, x(s)) ds - \int_0^t (t-s) u(s) x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-1} u(s) x(s) ds \right) \quad (3.3)$$

The first derivative and second derivative of (3.3)

$$x'(t) = x'(0) + tx''(0) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t f(s, x(s)) ds - \int_0^t u(s) x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^t (t-s)^{\alpha-2} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-2} u(s) x(s) ds \right) \quad (3.4)$$

and

$$x''(t) = -\frac{(3-\alpha)}{\beta(\alpha-2)} u(t) x(0) - \left(\frac{(3-\alpha)}{\beta(\alpha-2)}\right) t u(t) x'(0) + \left(1 - \left(\frac{(3-\alpha)}{\beta(\alpha-2)}\right) \frac{t^2}{2} u(t)\right) x''(0) + \frac{(3-\alpha)}{\beta(\alpha-2)} f(t, x(t)) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-2)} \left(\int_0^t (t-s)^{\alpha-3} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-3} u(s) x(s) ds \right) - \left(\frac{(3-\alpha)}{\beta(\alpha-2)}\right)^2 u(t) \left(\int_0^t (t-s)^{\alpha-2} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-2} u(s) x(s) ds \right) - \frac{(3-\alpha)(\alpha-2)}{(\beta(\alpha-2))^2 \Gamma(\alpha)} u(t) \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-1} u(s) x(s) ds \right). \quad (3.5)$$

Now, using periodic boundary conditions (PBCs) with necessary $f(0) = u(0)x(0)$, obtained that

$$x(0) = \frac{\alpha-2}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s) x(s) ds \right) + \frac{1}{u(T)} f(T, x(T)). \quad (3.6)$$

$$x'(0) = -\frac{(\alpha-2)}{3\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s) x(s) ds \right) + \frac{(\alpha-2)}{2\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s) x(s) ds \right) - \left(\frac{(3-\alpha)}{2\beta(\alpha-2)}\right) \left(\int_0^T (T-s) f(s, x(s)) ds - \int_0^T (T-s) u(s) x(s) ds \right) + \frac{(3-\alpha)}{2\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s) x(s) ds \right) \quad (3.7)$$

and

$$x''(0) = -\frac{(2+3A)(\alpha-2)}{6\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s) x(s) ds \right) + \frac{(3-\alpha)}{3\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s) x(s) ds \right). \quad (3.8)$$

Putting the values of $x(0)$, $x'(0)$ and $x''(0)$ in equation (3.3), we get

$$x(t) = -\left(\frac{t(t-T)}{2T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s) x(s) ds \right) \right) - \left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s) x(s) ds \right) \right) - \left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s) x(s) ds \right) \right) - \frac{1}{u(T)} f(T, x(T))$$

$$s)f(s, x(s))ds - \int_0^T (T-s)u(s)x(s)ds) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s))ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s)ds \right) - \left(\frac{t(t-T)}{2T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s))ds - \int_0^T u(s)x(s)ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s)f(s, x(s))ds - \int_0^t (t-s)u(s)x(s)ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds - \int_0^t (t-s)^{\alpha-1} u(s)x(s)ds \right) + \frac{1}{u(T)} f(T, x(T)) \quad u(t) \neq 0$$

After simplifications and replacing the value of $G(t, s)$ and $\mu(t, s)$

$$x(t) = \int_0^T G(t, s)f(s, x(s))ds + \int_0^T \mu(t, s)x(s)ds + \frac{1}{u(T)} f(T, x(T)).$$

This proves the theorem ■

In order to prove our point, we make the following assumptions.

Let $\psi_0 = C([0, T], \mathbb{R})$ denote the Banach space of all continuous functions $x: [0, T] \rightarrow \mathbb{R}$ with norm $\|x\| = \sup_{t \in [0, T]} |x(t)|$.

Assume that $\forall t \in [0, T]$ the following assumptions hold.

$$A_1 : |f(t, x_1(t)) - f(t, x_2(t))| < K|x_1 - x_2|. \quad K > 0$$

$$A_2 : |f(t, x(t))| \leq p(t)(1 + |x|). \quad p(t) \in \psi_0$$

$$A_3 : u(t) \text{ is bounded function, then there exists constant number } \lambda > 0 \text{ such that } |u(t)| \leq \lambda, \quad t \in [0, T]$$

$$A_4 : \Omega < 1, \quad \Omega := M_2.$$

$$M_2 := 2AT + 2CT^2 + \frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}.$$

3.1 Schauder's Fixed Point

In this section, we use Schauder's fixed point theorem. We start with the following theorem.

Theorem 3.4. [22]. (**Schauder's fixed point**) Let X be a Banach space, S be a bounded closed convex subset of X and $N: S \rightarrow S$ be a compact and continuous map. Then N has at least one fixed point in S .

Theorem 3.5. Assume that all assumptions (A_1) , (A_2) and (A_3) hold. Then the problem (1.1) – (2.2) has at least one solution on $[0, T]$

Proof. Consider $\beta_{r_2} = \{x \in \psi_0: \|x\| \leq r_2\}$ where $v \geq \max\{M_1(\|p\|)\}$ and

$$M_1 := \frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|}$$

$$r_2 \geq \frac{v}{1-(v+K_1)}.$$

$$K_1 := \frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}.$$

Now we defined operator N on β_{r_2}

$$(Nx)(t) = - \left(\frac{t(t-T)}{2T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s))ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s)ds \right) \right) - \left(\frac{t}{T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s))ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s)ds \right) \right) - \left(\frac{t}{T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s)f(s, x(s))ds - \int_0^T (T-s)u(s)x(s)ds \right) \right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s))ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s)ds \right) - \left(\frac{t(t-T)}{2T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s))ds - \int_0^T u(s)x(s)ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s)f(s, x(s))ds - \int_0^t (t-s)u(s)x(s)ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds - \int_0^t (t-s)^{\alpha-1} u(s)x(s)ds \right) + \frac{1}{u(T)} f(T, x(T)).$$

$$\|Nx\| = \sup_{t \in [0, T]} |(Nx)(t)|.$$

$$\|Nx\| = \left| -\left(\frac{t(t-T)}{2T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)}\left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T (T-s)f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds\right)\right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)}\left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s) ds\right) - \left(\frac{t(t-T)}{2T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds\right)\right) + \frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^t (t-s)f(s, x(s)) ds - \int_0^t (t-s)u(s)x(s) ds\right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-1} u(s)x(s) ds\right) + \frac{1}{u(T)} f(T, x(T)) \right|.$$

$$\|Nx\| \leq \left(\frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|}\right)(\|p\|(1+r_2)) + \left(\frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}\right)|r_2|.$$

$$\|Nx\| \leq M_1(\|p\|(1+r_2)) + K_1 r_2$$

$$\|Nx\| \leq v + (v + K_1) r_2$$

$$\|Nx\| \leq r_2$$

This shows that N is a self-mapping

Next, we'll go through two steps to prove that N has a fixed point using Schauder's fixed point theorem.

Step1: $N: \beta_{r_2} \rightarrow \beta_{r_2}$ is continuous

Let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence such that $x_n \rightarrow x$ in β_{r_2}

Then, for all $t \in [0, T]$, we have

$$\begin{aligned} |(Nx_n)(t) - (Nx)(t)| &= \left| -\left(\frac{t(t-T)}{2T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)}\left(\int_0^T (T-s)^{\alpha-2} f(s, x_n(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x_n(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^T (T-s)^{\alpha-1} f(s, x_n(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x_n(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T (T-s)f(s, x_n(s)) ds - \int_0^T (T-s) u(s)x_n(s) ds\right)\right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)}\left(\int_0^T (T-s)^{\alpha-3} f(s, x_n(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s)x_n(s) ds\right) - \left(\frac{t(t-T)}{2T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T f(s, x_n(s)) ds - \int_0^T u(s)x_n(s) ds\right)\right) + \frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^t (t-s)f(s, x_n(s)) ds - \int_0^t (t-s)u(s)x_n(s) ds\right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} f(s, x_n(s)) ds - \int_0^t (t-s)^{\alpha-1} u(s)x_n(s) ds\right) + \frac{1}{u(T)} f(T, x_n(T)) - \left(-\left(\frac{t(t-T)}{2T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)}\left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds\right)\right) - \left(\frac{t}{T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T (T-s)f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds\right)\right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)}\left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s) ds\right) - \left(\frac{t(t-T)}{2T}\right)\left(\frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds\right)\right) + \frac{(3-\alpha)}{\beta(\alpha-2)}\left(\int_0^t (t-s)f(s, x(s)) ds - \int_0^t (t-s)u(s)x(s) ds\right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s)^{\alpha-1} u(s)x(s) ds\right) + \frac{1}{u(T)} f(T, x(T)) \right| \\ |(Nx_n)(t) - (Nx)(t)| &\leq \left(\frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|}\right)K + \frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}(|x_n(t) - x(t)|). \end{aligned}$$

Therefore, obtained that

$$|(Nx_n)(t) - (Nx)(t)| \leq \lambda_1 |x_n(t) - x(t)|.$$

Where

$$\lambda_1 := \left(\frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|} \right) K + \frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}$$

Taking sup over $[0, T]$ we get,

$$\|Nx_n - Nx\| \leq \lambda_1 \|x_n - x\|.$$

Therefore, N is continuous.

Step2: $N(\beta_{r_2})$ is bounded and equicontinuous.

Since $N(\beta_{r_2}) \subset \beta_{r_2}$ and β_{r_2} is bounded. Then $N(\beta_{r_2})$ is bounded.

N it is continuous. Then we define $\sup_{t \in [0, T]} |f(t, x)| = D$

Let t_1 and t_2 belongs to $[0, T]$ and $x \in \beta_{r_2}$

$$\begin{aligned} |(Nx)(t_1) - (Nx)(t_2)| &= \left| - \left(\frac{t_1(t_1-T)}{2T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds \right) \right) - \right. \\ &\left. \left(\frac{t_1}{T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds \right) \right) - \right. \\ &\left. \left(\frac{t_1}{T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s) f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds \right) \right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s) ds \right) - \right. \\ &\left. \left(\frac{t_1(t_1-T)}{2T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^{t_1} (t_1-s) f(s, x(s)) ds - \int_0^{t_1} (t_1-s) u(s)x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^{t_1} (t_1-s)^{\alpha-1} f(s, x(s)) ds - \int_0^{t_1} (t_1-s)^{\alpha-1} u(s)x(s) ds \right) + \frac{1}{u(T)} f(T, x(T)) - \right. \\ &\left. \left(- \left(\frac{t_2(t_2-T)}{2T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds \right) \right) - \left(\frac{t_2}{T} \right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds \right) \right) - \right. \\ &\left. \left(\frac{t_2}{T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s) f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds \right) \right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-3} u(s)x(s) ds \right) - \right. \\ &\left. \left(\frac{t_2(t_2-T)}{2T} \right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^{t_2} (t_2-s) f(s, x(s)) ds - \int_0^{t_2} (t_2-s) u(s)x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^{t_2} (t_2-s)^{\alpha-1} f(s, x(s)) ds - \int_0^{t_2} (t_2-s)^{\alpha-1} u(s)x(s) ds \right) + \right. \\ &\left. \frac{1}{u(T)} f(T, x(T)) \right). \end{aligned}$$

Therefore, we find that

$$|(Nx)(t_1) - (Nx)(t_2)| \leq (S_1 + S_4)|t_1 - t_2| + (S_2 + S_3)|t_1^2 - t_2^2| + (S_5 + S_6)|t_1^\alpha - t_2^\alpha|.$$

$$S_1 := \left(\frac{\lambda(3-\alpha)T}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^{\alpha-1}}{2\beta(\alpha-2)\Gamma(\alpha+1)} \right) r_2, \quad S_2 := \left(\frac{\lambda(3-\alpha)}{\beta(\alpha-2)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{2\beta(\alpha-2)\Gamma(\alpha)} \right) r_2,$$

$$S_3 := \left(\frac{(3-\alpha)}{\beta(\alpha-2)} + \frac{(\alpha-2)T^{\alpha-2}}{2\beta(\alpha-2)\Gamma(\alpha)} \right) D, \quad S_4 := \left(\frac{(3-\alpha)T}{\beta(\alpha-2)} + \frac{(\alpha+2)(\alpha-2)T^{\alpha-1}}{2\beta(\alpha-2)\Gamma(\alpha+1)} \right) D,$$

$$S_5 := \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha+1)} \right) D \quad \text{and} \quad S_6 := \left(\frac{\lambda(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha+1)} \right) r_2$$

Which is independent of x and tends to zero as $t_2 \rightarrow t_1$ thus, N is relatively compact on β_{r_2} .

Hence, **by Arzela-Ascoli Theorem**, N is compact on β_{r_2} , thus all the assumptions of the theorem are satisfied. Hence the PBC's (1.1) – (1.2) has at least one solution on $[0, T]$.

3.2 Banach fixed point

In this part, we provide a unique solution to the problem (1.1) – (1.2). One of the most fundamental theorems is the Banach fixed point theorem. The most important aspect of any mathematical model is the unique solution; more than one solution might be meaningless and may not offer the necessary information. We begin by recalling Banach's fixed point theorem.

Theorem 3.6. [20] (Banach's fixed point theorem). Let ϱ be a non-empty closed subset of a Banach space X . Then any contraction mapping G of ϱ into itself has a unique fixed point.

Theorem 3.7. Assume that (A_1) and (A_3) , $\xi < 1$ hold. Then the problem (1.1) – (1.2) has a unique solution.

Where $\xi := (M_1K + M_2)$

Proof. Let us set $\sup_{t \in [0, T]} |f(t, 0)| \leq \eta$ and consider $\beta_r = \{x \in \psi_0 \mid \|x\| \leq r\}$, when $r \geq \frac{\xi_1}{1 - \xi_2}$ and

$$\xi_1 := \left(\frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|} \right) \eta, \quad \xi_2 := M_1K + M_2$$

We show that $H(\beta_r) \subset \beta_r$, where $H: \psi_0 \rightarrow \psi_0$

For $x \in \beta_r$, we have

$$\begin{aligned} \|Hx\| = & \left\| -\left(\frac{t(t-T)}{2T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds \right) - \right. \\ & \left. \left(\frac{t}{T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds \right) - \left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s) \right. \right. \right. \right. \\ & \left. \left. \left. f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds \right) \right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s) \right. \right. \\ & \left. \left. s)^{\alpha-3} u(s)x(s) ds \right) - \left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s) \right. \right. \\ & \left. \left. f(s, x(s)) ds - \int_0^t (t-s) u(s)x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s) \right. \right. \\ & \left. \left. s)^{\alpha-1} u(s)x(s) ds \right) + \frac{1}{u(T)} f(T, x(T)) \right\|. \end{aligned}$$

We consider

$$\begin{aligned} |f(t, x(t))| &= |f(t, x(t)) - f(t, 0) + f(t, 0)| \\ &\leq |f(t, x(t)) - f(t, 0)| + |f(t, 0)| \\ &\leq K|x| + \eta \end{aligned}$$

Therefore, obtained that

$$\|Hx\| \leq \left(\frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|} \right) (Kr + \eta) + M_2r$$

$$\|Hx\| \leq (M_1K + M_2)r + \xi_1 \leq r$$

H is a self-mapping on β_r .

Now for $x, y \in \beta_r$ and $\forall t \in [0, T]$, we consider

$$\begin{aligned} \|Hx - Hy\| = & \left\| -\left(\frac{t(t-T)}{2T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-2} u(s)x(s) ds \right) - \right. \\ & \left. \left(\frac{t}{T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, x(s)) ds - \int_0^T (T-s)^{\alpha-1} u(s)x(s) ds \right) - \left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s) \right. \right. \right. \right. \\ & \left. \left. \left. f(s, x(s)) ds - \int_0^T (T-s) u(s)x(s) ds \right) \right) \right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, x(s)) ds - \int_0^T (T-s) \right. \right. \\ & \left. \left. s)^{\alpha-3} u(s)x(s) ds \right) - \left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s)) ds - \int_0^T u(s)x(s) ds \right) \right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s) \right. \right. \\ & \left. \left. f(s, x(s)) ds - \int_0^t (t-s) u(s)x(s) ds \right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s)) ds - \int_0^t (t-s) \right. \right. \\ & \left. \left. s)^{\alpha-1} u(s)x(s) ds \right) + \frac{1}{u(T)} f(T, x(T)) \right\|. \end{aligned}$$

$$\begin{aligned}
& s)^{\alpha-3} u(s)x(s)ds) - \left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, x(s))ds - \int_0^T u(s)x(s)ds\right)\right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s) f(s, x(s))ds - \int_0^t (t-s) u(s)x(s)ds\right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds - \int_0^t (t-s)^{\alpha-1} u(s)x(s)ds\right) + \frac{1}{u(T)} f(T, x(T)) - \left(-\left(\frac{t(t-T)}{2T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha-1)} \left(\int_0^T (T-s)^{\alpha-2} f(s, y(s))ds - \int_0^T (T-s)^{\alpha-2} u(s)y(s)ds\right)\right) - \left(\frac{t}{T}\right) \left(\frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^T (T-s)^{\alpha-1} f(s, y(s))ds - \int_0^T (T-s)^{\alpha-1} u(s)y(s)ds\right)\right) - \left(\frac{t}{T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T (T-s) f(s, y(s))ds - \int_0^T (T-s) u(s)y(s)ds\right)\right) + \frac{(\alpha-2)}{u(T)(3-\alpha)\Gamma(\alpha-2)} \left(\int_0^T (T-s)^{\alpha-3} f(s, y(s))ds - \int_0^T (T-s)^{\alpha-3} u(s)y(s)ds\right) - \left(\frac{t(t-T)}{2T}\right) \left(\frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^T f(s, y(s))ds - \int_0^T u(s)y(s)ds\right)\right) + \frac{(3-\alpha)}{\beta(\alpha-2)} \left(\int_0^t (t-s) f(s, y(s))ds - \int_0^t (t-s) u(s)y(s)ds\right) + \frac{(\alpha-2)}{\beta(\alpha-2)\Gamma(\alpha)} \left(\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds - \int_0^t (t-s)^{\alpha-1} u(s)y(s)ds\right) + \frac{1}{u(T)} f(T, y(T)) \Bigg\|.
\end{aligned}$$

Hence, we get

$$\|Hx - Hy\| \leq (M_1K + M_2)\|x - y\|.$$

$$\|Hx - Hy\| \leq \xi\|x - y\|.$$

Since $\xi < 1$, by the Banach fixed point theorem, we get unique solution of (1.1) – (1.2). ■

4.0 CONCLUSION

In this section, our purpose is to find stability to the problem (1.1) – (1.2). We use the Hyers-Ulam stable. The notes and definitions that follow will be beneficial for our primary result.

Definition 4.1.[21]. The problem (1.1) – (1.2) is said to be Hyers-Ulam stable if there exists a real number $b > 0$ such that for each $\epsilon > 0$ and for each solution $\vartheta \in \psi_0$ of the inequality

$$|({}^{ABC}D_0^\alpha \vartheta)(t) + u(t)\vartheta(t) - f(t, \vartheta(t))| \leq \epsilon, \quad \forall t \in [0, T] \quad (4.1)$$

\exists a solution $x(t) \in \psi_0$ s.t

$$|\vartheta(t) - x(t)| \leq b\epsilon, \quad \forall t \in [0, T] \quad (4.2)$$

We can write the solution $x(t) \in \psi_0$ the problem of (1.1) – (1.2) as

$$x(t) = \int_0^T G(t, s)f(s, x(s))ds + \int_0^T \mu(t, s)x(s)ds + \frac{1}{u(T)}f(T, x(T))$$

Remark 4.2.

$$|\mu(t, s)| \leq \bar{b} \quad \text{and} \quad |G(t, s)| \leq M_f.$$

Remark 4.3. A function $\vartheta(t) \in \psi_0$ is a solution to inequality (4.1)

If and only if \exists a function $\zeta \in \psi_0$ such that

- I. $|\zeta(t)| \leq \epsilon$
- II. $({}^{ABC}D_0^\alpha \vartheta)(t) + \lambda \vartheta(t) = f(t, \vartheta(t)) + \zeta(t), \quad \forall t \in [0, T]$

Theorem 4.4. Suppose that $u, f: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. Satisfying Lipschitz condition (A_1) . If $(|u(T)|M_fTK + |u(T)|\bar{b}T + K) \neq |u(T)|$. Then the problem (1.1) – (1.2) is Hyers-Ulam stable.

Proof.

Let $x(t) \in \psi_0$ any solution to inequality

$$|({}^{ABC}D_0^\alpha x)(t) + u(t)x(t) - f(t, x(t))| \leq \epsilon, \quad \forall t \in [0, T]$$

Using **remark 4.3**, we have

$$({}^{ABC}D_0^\alpha x)(t) + u(t)x(t) = f(t, x(t)) + \zeta(t)$$

$$x(t) = \int_0^T G(t, s)f(s, x(s))ds + \int_0^T \mu(t, s)x(s)ds + \frac{1}{u(T)}f(T, x(T)) + \int_0^T G(t, s)\zeta(s)ds + \frac{1}{u(T)}\zeta(T)$$

$$\left| x(t) - \int_0^T G(t, s)f(s, x(s))ds - \int_0^T \mu(t, s)x(s)ds - \frac{1}{u(T)}f(T, x(T)) \right| \leq M_f T \epsilon + \frac{1}{|u(T)|} \epsilon.$$

Now let $y(t) \in \psi_0$ be a unique solution of fractional PBCs (1.1) – (1.2)

We consider

$$|x(t) - y(t)| = \left| x(t) - \int_0^T G(t, s)f(s, y(s))ds - \int_0^T \mu(t, s)y(s)ds - \frac{1}{u(T)}f(T, y(T)) \right|.$$

$$|x(t) - y(t)| = \left| x(t) - \int_0^T G(t, s)f(s, y(s))ds - \int_0^T \mu(t, s)y(s)ds - \frac{1}{u(T)}f(T, y(T)) + \right.$$

$$\left. \left(\int_0^T G(t, s)f(s, x(s))ds + \int_0^T \mu(t, s)x(s)ds + \frac{1}{u(T)}f(T, x(T)) \right) - \left(\int_0^T G(t, s)f(s, x(s))ds + \int_0^T \mu(t, s)x(s)ds + \frac{1}{u(T)}f(T, x(T)) \right) \right|.$$

$$|x(t) - y(t)| \leq M_f T \epsilon + \frac{1}{|u(T)|} \epsilon + \left| \int_0^T G(t, s)f(s, x(s))ds - \int_0^T G(t, s)f(s, y(s))ds + \int_0^T \mu(t, s)x(s)ds - \int_0^T \mu(t, s)y(s)ds + \frac{1}{u(T)}f(T, x(T)) - \frac{1}{u(T)}f(T, y(T)) \right|.$$

$$|x(t) - y(t)| \leq M_f T \epsilon + \frac{1}{|u(T)|} \epsilon + M_f T K |x(t) - y(t)| + \bar{b} T |x(t) - y(t)| + \frac{1}{|u(T)|} K |x(T) - y(T)|.$$

Taking sup over $t \in [0, T]$ or alternatively $\|x\| = \sup_{t \in [0, T]} |x(t)|$, we get

$$\|x - y\| \leq M_f T \epsilon + \frac{1}{|u(T)|} \epsilon + M_f T K \|x - y\| + \bar{b} T \|x - y\| + \frac{1}{|u(T)|} K \|x - y\|.$$

$$\|x - y\| \leq \frac{(|u(T)|M_f T + 1)\epsilon}{(|u(T)| - (|u(T)|M_f T K + |u(T)|\bar{b} T + K))}$$

$$b = \frac{(|u(T)|M_f T + 1)}{(|u(T)| - (|u(T)|M_f T K + |u(T)|\bar{b} T + K))}.$$

$$\|x - y\| \leq b \epsilon$$

Therefore the problem (1.1) – (1.2) is Hyers-Ulam stable. ■

Example. Consider the ABC-fractional PBCs

$$({}^{ABC}D_0^{2.1}x)(t) + \frac{1}{1000}(t+1)x(t) = f(t, x(t))$$

$$, 2 < \alpha < 3, t \in [0, 3], \text{ where } f(t, x(t)) = \frac{1}{1000+t} e^{-4t} x(t)$$

With periodic boundary conditions $x(0) = x(3), x'(0) = x'(3), x''(0) = x''(3)$.

Here obtain that, $\alpha = 2.1, T = 3$,

$$u(t) := \frac{1}{1000}(t+1), u(0) = \frac{1}{1000}, \lambda := \max |u(t)| = \frac{4}{1000},$$

$$f(t, x(t)) = \frac{1}{1000+t} e^{-4t} x(t), f(0) = \frac{1}{1000} x(0),$$

$$\text{Since } |f(t, x_1(t)) - f(t, x_2(t))| < \frac{1}{1000} |x_1 - x_2|, K = \frac{1}{1000}$$

$$M_1 := 302.4396, \Omega := M_2 = 0.2097 < 1,$$

$$\text{Where } M_1 := \frac{2(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{(2+\alpha)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)} + \frac{1}{|u(T)|}$$

$$M_2 := \frac{2\lambda(3-\alpha)T^2}{\beta(\alpha-2)} + \frac{\lambda(\alpha+2)(\alpha-2)T^\alpha}{\beta(\alpha-2)\Gamma(\alpha+1)} + \frac{\lambda(\alpha-2)T^{\alpha-2}}{|u(T)|(3-\alpha)\Gamma(\alpha-1)}$$

Therefore $\xi = M_1 K + M_2 = 0.5122 < 1$, $(|u(T)|M_f T K + |u(T)|\bar{b} T + K) \neq |u(T)|$, hence all conditions of the theorem 3.5 and theorem 3.7 are satisfy. As a result, the above PBCs are Hyers-Ulam stable and have a unique solution.

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