IJISCS (International Journal of Information System and Computer Science)

Full Paper
eISSN 2598-246X
pISSN 2598-0793

PERIODIC SOLUTION OF A SECOND ORDER OF DIFFERENTIAL EQUATIONS WITH HIGHER DERIVATIVES

*Corresponding author
raad.butris@uod.ac

Raad N. Butris[1], Noori R. Noori[2]
[1]Duhok University, College of basic education, Department of mathematics, Iraq.
[2]Duhok Polytechnic University, College of Engineering, Department of Energy, Iraq.
[1,2]Zakho Street 38, 1006 AJ Duhok Duhok, Kurdistan Region, Iraq.

Article history:
Received July 2, 2022
Revised August 26, 2022
Accepted August 29, 2022

Keywords:
Numerical-analytic method;
periodic differential equations;
higher derivatives;
Banach fixed point theorem.

Abstract
The study deals with the existence, uniqueness, and stability of periodic solution of a second order of differential equations with higher derivatives. We provide a wide range of qualifications including the numerical-analytic method has been used by the Samoilenko method to investigate the existence and approximation of periodic solutions of nonlinear systems of the differential equations. We give an appropriate solutions of the problem, and extend the results of Shlapak to more general cases by assuming the weaker conditions for the functions
$f\left(t,x,\frac{dx}{dt}\right)$ and $g\left(t,x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)$.

1.0 INTRODUCTION

Differential equations are essential for a mathematical description of nature, many of the general laws of nature-in physics, chemistry, biology, economics and engineering –find their most natural expression in the language of differential equation. Differential Equation (DE) allows us to study all kinds of evolutionary processes with the properties of determinacy; finite-dimensionality and differentiability. The study of DE began very soon after the invention of differential and integral calculus. In 1671, Newton had laid the foundation stone for the study of differential equations. He was followed by Leibnitz who coined the name differential equation in 1676 to denote relationship between differentials dx and dy of two variables x and y. The fundamental law of motion in mechanics, known as Newton's second law is a differential equation to describe the state of a system. Motion of a particle of mass m moving along a straight line under the influence of a specified external force
$f(t,x,x')$ is described by the following DE

$$mx' = f(t,x,x'), \qquad \left(x' = \frac{dx}{dt}, x'' = \frac{d^2x}{dt^2}\right)$$

At early stage, mathematicians were mostly engaged in formulating differential equations and solving them but they did not worry about the existence and uniqueness of solutions. **[1,6,7, 13,14]**. One of the most important theorems in ordinary differential equations is Picard's existence, uniqueness and stability theorem for first and second order of differential equations. A reason for this, can generalized to establish existence and uniqueness result for higher-order of differential equations and for a system of differential equations. Another is that, it is a good introduction to the broad class of existence, uniqueness and stability theorems that are based on fixed points. **[3,5,9,10,15,16]**. In recent years, Samoilenko assume that the numerical analytic method to study the periodic solution for ordinary differential equations and their algorithm structure. In the original work **[11].** The approach used and described here had been referred to as the numerical-analytic based upon successive approximations. The idea of the method, originally aimed at the investigation of periodic solution only. **[2,4,5,8,11,12,17]**.

Definition 1[8]. Let $(E, \|.\|)$ be a normed space. If T maps into itself we say that T is a contraction mapping on E if there exists $\alpha \in R$ with $0 < \alpha < 1$ such that:

$\|Tx - Ty\| \leq \alpha\|x - y\| \quad , \quad (x, y) \in E$

Lemma 1[8]. Let S be a space of all continuous functions on $[a, b]$, for any $t \in S$ we defined $\|z\| = \max_{t\in[a,b]} |z|$. Then $(S, \|.\|)$ is a Banach space.

Lemma 2[11]. Let the vector function $f(t, x)$ is defined and continuous on the interval$[0, T]$, then the following inequality $\left\| \int_0^t \left(f(s, x(s)) - \frac{1}{T}\int_0^T f(s, x(s))ds \right) ds \right\| \leq \alpha(t)M$

Is holds, where $\alpha(t) = 2t\left(1 - \frac{t}{T}\right)$ and $M = \max_{t\in[0,T]} |f(t, x)|$.

Definition2[8]. Let f be a continuous function define on a domain $G = \{(t, x): a \leq t \leq b, c \leq x \leq d\}$. Then f is said to satisfy a Lipschitz condition in the variable x on G, provided that a constant $L > 0$ exists with the property

$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2|$,

for all $(t, x_1), (t, x_2) \in G$. The constant L is called a Lipschitz constant for f .

Definition3[10]. Let f be a function on a set $E \subseteq R$. We say that f is lebesgue measurable on E or simply measurable if, for every $\alpha \in R$, the set $\{x: x \in E, f(x) > \alpha\}$ is measurable.

Definition4[10]. Let f be a Lebesgue measurable function defined on $E \subseteq R$. Let $L(E)$ be the set of all Lebesgue measurable functions defined on E such that:

$\int |f(x)|dx < \infty.$

The set $L(E)$ is called the set of Lebesgue measurable functions.

In [12,13] Shlapak studied the periodic solution of the differential equations which has the form:-

$\frac{dx}{dt} = g\left(t, x, \frac{dx}{dt}\right)$,

and

$\frac{d^2x}{dt^2} = g\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right)$

where $(t, x, \dot{x}, \ddot{x}) \in R^1 \times [a, b] \times [c, d] \times [e, f]$.

In this paper, we study the possibility of finding periodic solution of second order of differential equations with higher derivatives satisfied; let us consider this system of differential equations: -

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) + g\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}\right), \quad (1)$$

where $x \in D$, D is closed and bounded domain subset of R^n, D_1, D_2 are bounded domains subset of R^m.

Suppose that the vector functions $f(t, x, \dot{x})$ and $f(t, x, \dot{x}, \ddot{x})$ are defined and continuous on the domain:

$$(t, x, \dot{x}, \ddot{x}) \in R^1 \times D \times D_1 \times D_2 = (-\infty, \infty) \times D \times D_1 \times D_2 , \quad (2)$$

which are measurable functions and periodic in t of period T .

For any fixed $t \in [o, T]$ and let $m: D \to R^n$ such that $m \in L[0, T]$ satisfying the following inequalities:

$$\|f(t, x, \dot{x})\| \leq m(t), \|g(t, x, \dot{x}, \ddot{x})\| \leq n(t) \quad (3)$$

$$\|f(t, x_1, \dot{x}_1) - f(t, x_2, \dot{x}_2)\| \leq k_1(t)\|x_1 - x_2\| + k_2(t)\|\dot{x}_1 - \dot{x}_2\| \quad (4)$$

$\|g(t,x_1,\dot{x}_1,\ddot{x}_1)-g(t,x_2,\dot{x}_2,\ddot{x}_2)\|\leq\ l_1(t)\|x_1-x_2\|$
$+l_2(t)\|\dot{x}_1-\dot{x}_2\|+l_3(t)\|\ddot{x}_1-\ddot{x}_2\|$ (5)
for all $t\in R^1, x, x_1, x_2\in D,\ \dot{x},\dot{x}_1,\dot{x}_2\in D_1,\ \ddot{x},\ddot{x}_1,\ddot{x}_2\in D_2$, where $m(t), n(t), k_1(t), k_2(t), l_1(t), l_2(t)$ and $l_3(t)$ are integrable functions in $t\in[0,T]$ and $\dot{x}=\frac{dx}{dt}$, $\ddot{x}=\frac{d^2x}{dt^2}$

Define a non-empty sets as follows:-

$$\left.\begin{aligned} D_f &= D-\frac{T^2}{4}[\delta_1+\delta_2] \\ D_{1f} &= D_1-\frac{5T}{6}[\delta_1+\delta_2] \\ D_{2f} &= D_2-2[\|m(t)\|+\|n(t)\|] \end{aligned}\right\} \quad (6)$$

and the sequence of functions $\{x_m(t,x_0)\}_{m=0}^{\infty}$, defined as:

$$x_{m+1}(t,x_0)=x_0+L^2\left(f\big(t,x_m(t,x_0),\dot{x}_m(t,x_0)\big)+g\big(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\big)\right),$$
$$x(0,x_0)=x_0, m=0,1,2,\ldots, \quad (7)$$

where

$$\dot{x}_{m+1}(t,x_0)=\dot{x}_0+L\left(f\big(t,x_m(t,x_0),\dot{x}_m(t,x_0)\big)+g\big(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\big)\right)-$$
$$\frac{1}{T}\int_0^T L\left(f\big(t,x_m(t,x_0),\dot{x}_m(t,x_0)\big)+g\big(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\big)\right)dt, \quad (8)$$

$\dot{x}(0,x_0)=\dot{x}_0$, m= 0,1,2,…

and

$$\ddot{x}_{m+1}(t,x_0)=\ddot{x}_0+f\big(t,x_m(t,x_0),\dot{x}_m(t,x_0)\big)+g\big(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\big)-$$
$$\frac{1}{T}\int_0^T\left(f\big(t,x_m(t,x_0),\dot{x}_m(t,x_0)\big)+g\big(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\big)\right)dt \quad (9)$$

with

$\ddot{x}(0,x_0)=\ddot{x}_0, m=0,1,2,\ldots.$

The operator L is defined by the equation $Lf(t)=\int_0^t\left(f(s)-\frac{1}{T}\int_0^T f(t)dt\right)ds$ where $f(t)=f\left(t,x,\frac{dx}{dt}\right)+g\left(t,x,\frac{dx}{dt},\frac{d^2x}{dt^2}\right)$.

2.0 THEORETICAL

2.0. Approximate periodic solution of (1)

In this section, we study the approximate of periodic solution for the problem (1) by the following theorem:

Theorem1. Let the vector functions $f(t,x,\dot{x})$ and $g(t,x,\dot{x},\ddot{x})$ measurable in t for any fixed point $x\in[0,T]$and bounded by Lebesgue integrable functions $m(t)$ and $n(t)$.Then there exist a sequence of functions (7) is periodic in t of period T, converges uniformly as $m\to\infty$ in the domain:

$$(t,x_0)\in[0,T]\times D_f \quad (10)$$

to the limit function $x(t,x_0)$ and satisfy the following integral equations:

$$x(t,x_0)=x_0+L^2\left(f\big(t,x(t,x_0),\dot{x}(t,x_0)\big)+g\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)\right), \quad (11)$$

where

$$\dot{x}(t,x_0)=\dot{x}_0+L\left(f\big(t,x(t,x_0),\dot{x}(t,x_0)\big)+g\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)\right)-\frac{1}{T}\int_0^T L\left(f\big(t,x(t,x_0),\dot{x}(t,x_0)\big)+g\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)\right)dt \quad \ldots(12)$$

and

$$\ddot{x}(t,x_0)=\ddot{x}_0+f\big(t,x(t,x_0),\dot{x}(t,x_0)\big)+g\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)-\frac{1}{T}\int_0^T\left(f\big(t,x(t,x_0),\dot{x}(t,x_0)\big)+g\big(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\big)\right)dt \quad \ldots(13)$$

which is a periodic solution of the problem (1) provided that:

$$\left.\begin{aligned} \|x^0(t,x_0)-x_0\| &\le \tfrac{T^2}{4}[\delta_1+\delta_2] \\ \|x^0(t,x_0)-x_m(t,x_0)\| &\le Q^m(E-Q)^{-1}\,\tfrac{T^2}{4}[\delta_1+\delta_2] \end{aligned}\right\} \qquad (14)$$

Proof. By the relations (7),(8) ,(9) and lemma 1, we get

$$\begin{pmatrix} \|x_1(t,x_0)-x_0\| \\ \|\dot{x}_1(t,x_0)-\dot{x}_0\| \\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\| \end{pmatrix} \le \begin{pmatrix} \left\|L\left(L\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)\right)\right\| \\ \left\|L\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)-\dfrac{1}{T}\displaystyle\int_0^T L\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)dt\right\| \\ \left\|2\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)\right\| \end{pmatrix}$$

Choosing $\int_0^T m(s)ds \le \delta_1$ and $\int_0^T n(s)ds \le \delta_2$, $\delta_1,\delta_2 > 0$.
So that

$$\begin{pmatrix} \|x_1(t,x_0)-x_0\| \\ \|\dot{x}_1(t,x_0)-\dot{x}_0\| \\ \|\ddot{x}_1(t,x_0)-\ddot{x}_0\| \end{pmatrix} \le \begin{pmatrix} \frac{T^2}{4}[\delta_1+\delta_2] \\ \left(\frac{T}{2}+\frac{T}{3}\right)[\delta_1+\delta_2] \\ 2[\|m(t)\|+\|n(t)\|] \end{pmatrix}$$

Therefore, $x_1(t,x_0)\in D$, $\dot{x}_1(t,x_0)\in D_1$ and $\ddot{x}_1(t,x_0)\in D_2$, for all $t\in[0,T]$.
Then, by mathematical induction we can prove that:-

$$\begin{pmatrix} \|x_m(t,x_0)-x_0\| \\ \|\dot{x}_m(t,x_0)-\dot{x}_0\| \\ \|\ddot{x}_m(t,x_0)-\ddot{x}_0\| \end{pmatrix} \le \begin{pmatrix} \frac{T^2}{4}[\delta_1+\delta_2] \\ \frac{5T}{6}[\delta_1+\delta_2] \\ 2[\|m(t)\|+\|n(t)\|] \end{pmatrix} \qquad (15)$$

i. e. $x_m(t,x_0)\in D$, $\dot{x}_m(t,x_0)\in D_1$ and $\ddot{x}_m(t,x_0)\in D_2$ when $x_0\in D_f$, $\dot{x}_0\in D_{1f}$ and $\ddot{x}_0\in D_{2f}$.

Next, we prove that the sequence of function (7) convergent uniformly on the domain (2). When $m=1$, we get

$$\begin{pmatrix} \|x_2(t,x_0)-x_1(t,x_0)\| \\ \|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\| \\ \|\ddot{x}_2(t,x_0)-\ddot{x}_1(t,x_0)\| \end{pmatrix} =$$

$$\begin{pmatrix} \left\| \begin{matrix} x_0+\displaystyle\int_0^t L\begin{pmatrix} f\big(s,x_1(s,x_0),\dot{x}_1(s,x_0)\big)+g\big(s,x_1(s,x_0),\dot{x}_1(s,x_0),\ddot{x}_1(s,x_0)\big)- \\ \dfrac{1}{T}\displaystyle\int_0^T\Big(f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0)\big)+g\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)\Big)dt \end{pmatrix}ds \\ -x_0-\displaystyle\int_0^t L\left(f(s,x_0,\dot{x}_0)+g(s,x_0,\dot{x}_0,\ddot{x}_0)-\dfrac{1}{T}\int_0^T\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)dt\right)ds \end{matrix} \right\| \\ \left\| \begin{matrix} \dot{x}_0+L\begin{pmatrix} f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0)\big)+g\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)- \\ \dfrac{1}{T}\displaystyle\int_0^T\Big(f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0)\big)+g\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)\Big)dt \end{pmatrix} \\ -\dot{x}_0-L\left(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)-\dfrac{1}{T}\displaystyle\int_0^T\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)dt\right) \end{matrix} \right\| \\ \left\| \begin{matrix} \ddot{x}_0+f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0)\big)+g\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)- \\ \dfrac{1}{T}\displaystyle\int_0^T\Big(f\big(t,x_1(t,x_0),\dot{x}_1(t,x_0)\big)+g\big(t,x_1(t,x_0),\dot{x}_1(t,x_0),\ddot{x}_1(t,x_0)\big)\Big)dt \\ -\ddot{x}_0-f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)-\dfrac{1}{T}\displaystyle\int_0^T\big(f(t,x_0,\dot{x}_0)+g(t,x_0,\dot{x}_0,\ddot{x}_0)\big)dt \end{matrix} \right\| \end{pmatrix}$$

here $\int_0^T K_1(s)ds\le\mu_1$, $\int_0^T K_2(s)ds\le\mu_2$, $\int_0^T L_1(s)ds\le\sigma_1$, $\int_0^T L_2(s)ds\le\sigma_2$, $\int_0^T L_3(s)ds\le\sigma_3$ and $\mu_1,\mu_2,\sigma_1,\sigma_2,\sigma_3,K_1,K_2,L_1,L_2,L_3>0$.

$$\begin{pmatrix}\|x_2(t,x_0)-x_1(t,x_0)\|\\\|\dot{x}_2(t,x_0)-\dot{x}_1(t,x_0)\|\\\|\ddot{x}_2(t,x_0)-\ddot{x}_1(t,x_0)\|\end{pmatrix}\le$$

$$\begin{pmatrix}\frac{T^2}{4}\left((\mu_1+\sigma_1)\|x_1(t,x_0)-x_0\|+(\mu_2+\sigma_2)\|\dot{x}_1(t,x_0)-\dot{x}_0\|+\sigma_3\|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\right)\\\left(\alpha(t)+\frac{T}{3}\right)\left((\mu_1+\sigma_1)\|x_1(t,x_0)-x_0\|+(\mu_2+\sigma_2)\|\dot{x}_1(t,x_0)-\dot{x}_0\|+\sigma_3\|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\right)\\2\left((\mu_1+\sigma_1)\|x_1(t,x_0)-x_0\|+(\mu_2+\sigma_2)\|\dot{x}_1(t,x_0)-\dot{x}_0\|+\sigma_3\|\ddot{x}_1(t,x_0)-\ddot{x}_0\|\right)\end{pmatrix}$$

Therefore

$$\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\|\\\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|\\\|\ddot{x}_{m+1}(t,x_0)-\ddot{x}_m(t,x_0)\|\end{pmatrix}=$$

$$\begin{pmatrix}\left\|\begin{matrix}x_0+\int_0^t L\begin{pmatrix}f(s,x_m(s,x_0),\dot{x}_m(s,x_0))+g(s,x_m(s,x_0),\dot{x}_m(s,x_0),\ddot{x}_m(s,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_m(t,x_0),\dot{x}_m(t,x_0))+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))dt\right)\end{pmatrix}ds-x_0-\\\int_0^t L\begin{pmatrix}f(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0))+g(s,x_{m-1}(s,x_0),\dot{x}_{m-1}(s,x_0),\ddot{x}_{m-1}(s,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0))+g(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))\right)dt\end{pmatrix}ds\end{matrix}\right\|\\\left\|\begin{matrix}\dot{x}_0+L\begin{pmatrix}f(t,x_m(t,x_0),\dot{x}_m(t,x_0))+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_m(t,x_0),\dot{x}_m(t,x_0))+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))\right)dt\end{pmatrix}-\\\dot{x}_0-L\begin{pmatrix}f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0))+g(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0))+g(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))\right)dt\end{pmatrix}\end{matrix}\right\|\\\left\|\begin{matrix}\ddot{x}_0+f(t,x_m(t,x_0),\dot{x}_m(t,x_0))+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_m(t,x_0),\dot{x}_m(t,x_0))+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))dt\right)-\\\ddot{x}_0-f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0))+g(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))-\\\frac{1}{T}\int_0^T\left(f(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0))+g(t,x_{m-1}(t,x_0),\dot{x}_{m-1}(t,x_0),\ddot{x}_{m-1}(t,x_0))\right)dt\end{matrix}\right\|\end{pmatrix}$$

$$\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\|\\\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|\\\|\ddot{x}_{m+1}(t,x_0)-\ddot{x}_m(t,x_0)\|\end{pmatrix}\le$$

$$\begin{pmatrix}\alpha(t)\frac{T}{2}\begin{pmatrix}(\mu_1+\sigma_1)\|x_m(t,x_0)-x_{m-1}(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}_m(t,x_0)-\dot{x}_{m-1}(t,x_0)\|\\+\sigma_3\|\ddot{x}_m(t,x_0)-\ddot{x}_{m-1}(t,x_0)\|\end{pmatrix}\\\left(\alpha(t)+\frac{T}{3}\right)\begin{pmatrix}(\mu_1+\sigma_1)\|x_m(t,x_0)-x_{m-1}(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}_m(t,x_0)-\dot{x}_{m-1}(t,x_0)\|\\+\sigma_3\|\ddot{x}_m(t,x_0)-\ddot{x}_{m-1}(t,x_0)\|\end{pmatrix}\\2\begin{pmatrix}((\mu_1+\sigma_1))\|x_m(t,x_0)-x_{m-1}(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}_m(t,x_0)-\dot{x}_{m-1}(t,x_0)\|\\+\sigma_3\|\ddot{x}_m(t,x_0)-\ddot{x}_{m-1}(t,x_0)\|\end{pmatrix}\end{pmatrix}$$

Hence

$$\begin{pmatrix}\|x_{m+1}(t,x_0)-x_m(t,x_0)\|\\\|\dot{x}_{m+1}(t,x_0)-\dot{x}_m(t,x_0)\|\\\|\ddot{x}_{m+1}(t,x_0)-\ddot{x}_m(t,x_0)\|\end{pmatrix}\le$$

$$\begin{pmatrix}\alpha(t)\frac{T}{2}(\mu_1+\sigma_1) & \alpha(t)\frac{T}{2}(\mu_2+\sigma_2) & \alpha(t)\frac{T}{2}\sigma_3\\\left(\alpha(t)+\frac{T}{3}\right)(\mu_1+\sigma_1) & \left(\alpha(t)+\frac{T}{3}\right)(\mu_2+\sigma_2) & \left(\alpha(t)+\frac{T}{3}\right)\sigma_3\\2(\mu_1+\sigma_1) & 2(\mu_2+\sigma_2) & 2\sigma_3\end{pmatrix}\begin{pmatrix}\|x_m(t,x_0)-x_{m-1}(t,x_0)\|\\\|\dot{x}_m(t,x_0)-\dot{x}_{m-1}(t,x_0)\|\\\|\ddot{x}_m(t,x_0)-\ddot{x}_{m-1}(t,x_0)\|\end{pmatrix}$$

$$(16)$$

Rewrite inequality (16) by the following form:

$$z_{m+1}(t) \le Q(t) z_m^0 \qquad (17)$$

where

$$z_{m+1}(t) = \begin{pmatrix} \|x_{m+1}(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}_{m+1}(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}_{m+1}(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix}$$

$$Q(t) = \begin{pmatrix} \alpha(t)\frac{T}{2}(\mu_1+\sigma_1) & \alpha(t)\frac{T}{2}\ (\mu_2+\sigma_2) & \alpha(t)\frac{T}{2}\sigma_3 \\ \left(\alpha(t)+\frac{T}{3}\right)(\mu_1+\sigma_1) & \left(\alpha(t)+\frac{T}{3}\right)(\mu_2+\sigma_2) & \left(\alpha(t)+\frac{T}{3}\right)\sigma_3 \\ 2(\mu_1+\sigma_1) & 2(\mu_2+\sigma_2) & 2\sigma_3 \end{pmatrix}$$

$$z_m^0 = \begin{pmatrix} \|x_m(t,x_0) - x_{m-1}(t,x_0)\| \\ \|\dot{x}_m(t,x_0) - \dot{x}_{m-1}(t,x_0)\| \\ \|\ddot{x}_m(t,x_0) - \ddot{x}_{m-1}(t,x_0)\| \end{pmatrix}$$

$$z_1^0 \le \begin{pmatrix} \frac{T^2}{4}[\delta_1+\delta_2] \\ \frac{5T}{6}[\delta_1+\delta_2] \\ 2[\|m(t)\| + \|n(t)\|] \end{pmatrix}$$

Taking the max. of two sides of the inequality (17, we find that

$$z_{m+1}^0 \le Q_0\, z_m^0 \qquad (18)$$

Where

$$Q_0 = \begin{pmatrix} \frac{T^2}{4}(\mu_1+\sigma_1) & \frac{T^2}{4}\ (\mu_2+\sigma_2) & \frac{T^2}{4}\sigma_3 \\ \frac{5T}{6}(\mu_1+\sigma_1) & \frac{5T}{6}(\mu_2+\sigma_2) & \frac{5T}{6}\sigma_3 \\ 2(\mu_1+\sigma_1) & 2(\mu_2+\sigma_2) & 2\sigma_3 \end{pmatrix} \qquad (19)$$

By iterating inequality (18) we have

$$z_{m+1}^0 \le Q_0^m\, z_1^0 \qquad (20)$$

which leads to the estimate

$$\sum_{i=1}^{m} z_i^0 \le \sum_{i=1}^{m} Q_0^{i-1}\, z_1^0 \qquad (21)$$

Since the matrix Q_0 has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \frac{T^2}{4}(\mu_1+\sigma_1) + \frac{5T}{6}(\mu_2+\sigma_2) + 2\sigma_3 < 1$, the series (21) is uniformly convergent, that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} Q_0^{i-1} = \sum_{i=1}^{\infty} Q_0^{i-1}\, z_1^0 = (E - Q_0)^{-1} z_1^0 \qquad (22)$$

The limiting relation (22) signifies a uniform convergence of the sequence (7), $i.e$

$$\lim_{m\to\infty} x_m(t,x_0) = x^0(t,x_0) \qquad (23)$$

where

$$\lim_{m\to\infty} \dot{x}_m(t,x_0) = \dot{x}^0(t,x_0)$$

and

$$\lim_{m\to\infty} \ddot{x}_m(t,x_0) = \ddot{x}^0(t,x_0)$$

By inequality (17), the estimate

$$\begin{pmatrix} \|x^0(t,x_0) - x_m(t,x_0)\| \\ \|\dot{x}^0(t,x_0) - \dot{x}_m(t,x_0)\| \\ \|\ddot{x}^0(t,x_0) - \ddot{x}_m(t,x_0)\| \end{pmatrix} \le Q_0^m (E - Q_0)^{-1} z_1^0\ , \qquad (24)$$

for all m≥ 0.

Using relation (23) and proceeding in equation (7) to the limit $m \to \infty$, convince us that the function $x^0(t,x_0)$ is the periodic solution of (1).

4.0 RESULANTS

3.0. Existence of a periodic solution of (1)

The problem of the existence of a periodic solution of a period T of equation (1) is uniquely connected with that of the existence of zeros of the function $\Delta(t,x_0)$ which has the form:

$$\Delta(0,x_0)=\frac{1}{T}\int_0^T\left[f\left(t,x(t,x_0),\dot{x}(t,x_0)\right)+g(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0))\right]dt \quad (25)$$

Since the function $\Delta(0,x_0)$ is found only approximately, from the sequence of functions

$$\Delta_{\mathrm{m}}(0,x_0)=\frac{1}{T}\int_0^T\left[f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0)\right)+g(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0))\right]dt \quad (26)$$

where $m=0,1,2,\dots$

Theorem2. Under the hypothesis and conditions of theorem1, the following inequality:

$\|\Delta(t,x_0)-\Delta_m(t,x_0)\|\le d_m$

is hold for all $m\ge 0$,

where

$$d_m=\left\langle\begin{pmatrix}\mu_1+\sigma_1\\ \mu_2+\sigma_2\\ \sigma_3\end{pmatrix},Q_0^m(E-Q_0)^{-1}z_1^0\right\rangle. \quad (27)$$

Proof. Assuming

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\|=\left\|\frac{1}{T}\int_0^T[f(t,x^0(t,x_0),\dot{x}^0(t,x_0))+g(t,x^0(t,x_0),\dot{x}^0(t,x_0),\ddot{x}^0(t,x_0))]dt-\frac{1}{T}\int_0^T\left[f\left(t,x_m(t,x_0),\dot{x}_m(t,x_0)\right)+g\left(t,x_m(t,x_0),\dot{x}_m(t,x_0),\ddot{x}_m(t,x_0)\right)\right]dt\right\|$$

So

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\|\le(\mu_1+\sigma_1)\|x^0(t,x_0)-x_m(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}^0(t,x_0)-\dot{x}_m(t,x_0)\|+\sigma_3\|\ddot{x}^0(t,x_0)-\ddot{x}_m(t,x_0)\|$$

Hence

$$\|\Delta(0,x_0)-\Delta_m(0,x_0)\|\le\left\langle\begin{pmatrix}\mu_1+\sigma_1\\ \mu_2+\sigma_2\\ \sigma_3\end{pmatrix},Q_0^m(E-Q_0)^{-1}\,z_1^0\right\rangle=d_m$$

for all $m\ge 0$.

Remark1 [11]. When R^n=R^1, i.e. when x is a scalar the following theorem can be strengthens by giving up the requirement that the singular point shout be isolated, thus we have:

Theorem3. Let the problem (1) be defined on an interval $a\le x\le b$ of a straight line R^1. Assume that for a real t and an integer $m\ge 1$, the function (26) satisfies the inequalities

$$\left.\begin{aligned}\min_{a+\frac{T^2}{4}[\delta_1+\delta_2]\le x\le b-\frac{T^2}{4}[\delta_1+\delta_2]}\Delta_{\mathrm{m}}(x)\le -d_m\\ \max_{a+\frac{T^2}{4}[\delta_1+\delta_2]\le x\le b-\frac{T^2}{4}[\delta_1+\delta_2]}\Delta_{\mathrm{m}}(x)\ge d_m\end{aligned}\right\} \quad (28)$$

where $d_m=\left\langle\begin{pmatrix}\mu_1+\sigma_1\\ \mu_2+\sigma_2\\ \sigma_3\end{pmatrix},Q_0^m(E-Q_0)^{-1}z_1^0\right\rangle$. Then the problem (1) has periodic solution $x=x(t,x_0)$ for which $a+\frac{T^2}{4}[\delta_1+\delta_2]\le x\le b-\frac{T^2}{4}[\delta_1+\delta_2]$.

Proof. Let x_1 and x_2 be points of the interval $[a+\frac{T^2}{4}[\delta_1+\delta_2]\le x\le b-\frac{T^2}{4}[\delta_1+\delta_2]]$ such that

$\Delta_{\mathrm{m}}(x_1)=\min\Delta_{\mathrm{m}}(x)\,,\Delta_{\mathrm{m}}(x_2)=\max\Delta_{\mathrm{m}}(x)$

$$x\in\left(a+\frac{T^2}{4}[\delta_1+\delta_2],b-\frac{T^2}{4}[\delta_1+\delta_2]\right)$$

From the inequalities (27) and (28), we have

$$\left.\begin{aligned}\Delta(x_1)=\Delta_m(x_1)+\left(\Delta(x_1)-\Delta_m(x_1)\right)\le 0\\ \Delta(x_2)=\Delta_m(x_2)+\left(\Delta(x_2)-\Delta_m(x_2)\right)\ge 0\end{aligned}\right\} \quad (29)$$

It follows from (29) in virtue of the continuity of the Δ-constant that there exist a point $x^0, x^0 \in [x_1, x_2]$ such that $\Delta(x^0) = 0$. This means that the system (1) has a periodic solution $x = x(t, x_0)$.

Theorem4. Let a system (1) be given in the domain D. Suppose that D_1 is a set belonging D_f .then, for D_1 to have a point at which the Δ-constant is zero, it is necessary that, for some t, all integral m's , and any $x_1 \in D_1$, the following integrality hold true:

$$\|\Delta_m(t,x)\| \le (\delta_1 + \delta_2) + \left\langle \begin{pmatrix} \mu_1 + \sigma_1 \\ \mu_2 + \sigma_2 \\ \sigma_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} z_1^0 \right\rangle$$

for all $m \ge 0$.

Proof. Let the Δ-constant at the point $x \in D_1$ be zero

Since,

$$\|\Delta_m(0, x_1)\| = \|\Delta_m(0, x_1) - \Delta(0, x_1) + \Delta(0, x_1)\|$$

Then

$$\|\Delta_m(0, x_1)\| \le \|\Delta(0, x_1)\| + \|\Delta_m(0, x_1) - \Delta(0, x_1)\|$$

and hence

$$\|\Delta_m(0, x_1)\| \le \|\Delta(0, x_1)\| + \left\langle \begin{pmatrix} \mu_1 + \sigma_1 \\ \mu_2 + \sigma_2 \\ \sigma_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} z_1^0 \right\rangle$$

But

$$\|\Delta_m(0, x_1)\| \le \left\| \frac{1}{T} \int_0^T [f(t, x_m(t, x_0), \dot{x}_m(t, x_0)) + g(t, x_m(t, x_0), \dot{x}_m(t, x_0), \ddot{x}_m(t, x_0))] dt \right\| + \left\langle \begin{pmatrix} \mu_1 + \sigma_1 \\ \mu_2 + \sigma_2 \\ \sigma_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} z_1^0 \right\rangle$$

Therefore

$$\|\Delta_m(t, x_1)\| \le (\delta_1 + \delta_2) + \left\langle \begin{pmatrix} \mu_1 + \sigma_1 \\ \mu_2 + \sigma_2 \\ \sigma_3 \end{pmatrix}, Q_0^m (E - Q_0)^{-1} z_1^0 \right\rangle.$$

for all $m \ge 0$.

4.0 RESULANTS

4.0. Another Method

In this section, we also proving the existence and uniqueness theorem for (1) by using Banach fixed point theorem.

Theorem5. Let the vector functions $f(t, x, \dot{x})$ and $g(t, x, \dot{x}, \ddot{x})$ be defined, measurable on the domain (2) and satisfy the assumptions and conditions of theorem1.Then the problem (1) has a unique periodic solution on the domain (2).

Proof. Let $(C[0,T], \|.\|)$ Be a Banach space and T^* be a mapping on $C[0,T]$ as follows:

$$T^* x(t, x_0) = x_0 + L^2 \Big(f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) \Big)$$

where

$$T^* \dot{x}(t, x_0) = \dot{x}_0 + L \Big(f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) \Big) - \frac{1}{T} \int_0^T L \Big(f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) \Big) dt$$

and

$$T^* \ddot{x}(t, x_0) = \ddot{x}_0 + f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) - \frac{1}{T} \int_0^T \Big(f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) \Big) dt$$

Since $\int_0^t \Big(L \Big(f(s, x(s, x_0), \dot{x}(s, x_0)) + g(s, x(s, x_0), \dot{x}(s, x_0), \ddot{x}(s, x_0)) \Big) - \frac{1}{T} \int_0^T L \Big(f(t, x(t, x_0), \dot{x}(t, x_0)) + g(t, x(t, x_0), \dot{x}(t, x_0), \ddot{x}(t, x_0)) \Big) dt \Big) ds$ is continuous on the domain (2).

And also

$L\left(f\left(t,x(t,x_0),\dot{x}(t,x_0)\right)+g\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)\right)-\frac{1}{T}\int_0^T L\left(f\left(t,x(t,x_0),\dot{x}(t,x_0)\right)+g\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)\right)dt$, $f\left(t,x(t,x_0),\dot{x}(t,x_0)\right)+g\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)-\frac{1}{T}\int_0^T\left(f\left(t,x(t,x_0),\dot{x}(t,x_0)\right)+g\left(t,x(t,x_0),\dot{x}(t,x_0),\ddot{x}(t,x_0)\right)\right)dt$ are continuous on the same domain (2).

Therefore: $T^*: C[0,T] \longrightarrow C[0,T]$

Now, we shall to prove that T^* is a contraction mapping on $[0,T]$.

Let $x(t,x_0), z(t,x_0)$ be a vector functions on $[0,T]$, then

$$\|T^*x(t,x_0)-T^*z(t,x_0)\| = \max_{t\in[0,T]}\{|T^*x(t,x_0)-T^*z(t,x_0)|\}$$

and

$$\|T^*\dot{x}(t,x_0)-T^*\dot{z}(t,x_0)\| = \max_{t\in[0,T]}\{|T^*\dot{x}(t,x_0)-T^*\dot{z}(t,x_0)|\}$$

and also

$$\|T^*\ddot{x}(t,x_0)-T^*\ddot{z}(t,x_0)\| = \max_{t\in[0,T]}\{|T^*\ddot{x}(t,x_0)-T^*\ddot{z}(t,x_0)|\}$$

So

$$\begin{pmatrix}\|T^*x(t,x_0)-T^*z(t,x_0)\|\\ \|T^*\dot{x}(t,x_0)-T^*\dot{z}(t,x_0)\|\\ \|T^*\ddot{x}(t,x_0)-T^*\ddot{z}(t,x_0)\|\end{pmatrix} \le \begin{pmatrix}\alpha(t)\frac{T}{2}((\mu_1+\sigma_1)\|x(t,x_0)-z(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|+\sigma_3\|\ddot{x}(t,x_0)-\ddot{z}(t,x_0)\|)\\ \left(\alpha(t)+\frac{T}{3}\right)((\mu_1+\sigma_1)\|x(t,x_0)-z(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|+\sigma_3\|\ddot{x}(t,x_0)-\ddot{z}(t,x_0)\|)\\ 2((\mu_1+\sigma_1)\|x(t,x_0)-z(t,x_0)\|+(\mu_2+\sigma_2)\|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|+\sigma_3\|\ddot{x}(t,x_0)-\ddot{z}(t,x_0)\|)\end{pmatrix}$$

$$\begin{pmatrix}\|T^*x(t,x_0)-T^*z(t,x_0)\|\\ \|T^*\dot{x}(t,x_0)-T^*\dot{z}(t,x_0)\|\\ \|T^*\ddot{x}(t,x_0)-T^*\ddot{z}(t,x_0)\|\end{pmatrix} \le \begin{pmatrix}\alpha(t)\frac{T}{2}(\mu_1+\sigma_1) & \alpha(t)\frac{T}{2}(\mu_2+\sigma_2) & \alpha(t)\frac{T}{2}\sigma_3\\ \left(\alpha(t)+\frac{T}{3}\right)(\mu_1+\sigma_1) & \left(\alpha(t)+\frac{T}{3}\right)(\mu_2+\sigma_2) & \left(\alpha(t)+\frac{T}{3}\right)\sigma_3\\ 2(\mu_1+\sigma_1) & 2(\mu_2+\sigma_2) & 2\sigma_3\end{pmatrix}\begin{pmatrix}\|x(t,x_0)-z(t,x_0)\|\\ \|\dot{x}(t,x_0)-\dot{z}(t,x_0)\|\\ \|\ddot{x}(t,x_0)-\ddot{z}(t,x_0)\|\end{pmatrix}$$

By the condition $\lambda_{max}\Lambda < 1$, then T^* is a contraction mapping on $C[0,T]$.

Thus, by Banach fixed point theorem then there exists a fixed point $x(t,x_0)$ in $C[0,T]$ such that $T^*x(t,x_0)=x(t,x_0)$

Therefore the integral equation (11) is a unique solution of (1).

5.0 CONCLUSION

This paper provided the existence and approximation of the periodic solutions for nonlinear system of a second order of differential equations with higher derivatives. Theorems on existence and uniqueness of periodic solution are established under some necessary and sufficient conditions on closed and bounded domains (compact spaces). The numerical-analytic method has been used to study the periodic solutions of ordinary differential equations which were introduced by Samoilenko.

REFERENCES

[1] Aziz, M. A. (2006). Periodic solutions for some systems of nonlinear ordinary differential equations. Thesis. Collage of education. University of Mosul.

[2] Butris, R. N. and Abdullah, D. S. (2015). Solution of Integro-differential Equation of the first order with the operators. International Journal of Scientific Engineering and Applied Science. India (IJSEAS).Volume1.Issue 6

[3] Goma, I. A. (1977). Method of successive approximation in a two-point boundary value problem with parameter. Ukrainskii matematicheskii zhurnal. 29 (6). 800-807.

[4] Hu, J. and Li, W. P. (2005). Theory of ordinary differential equations. The Hong Kong University of Science and Technology.

[5] Korol, I. I. (2005). On periodic solutions of one class of systems of differential equations. Ukrainian mathematical journal. Vol. 57. No. 4.

[6] Lipshutz, D. and Williams, R. J. (2015). Existence, Uniqueness, and Stability of Slowly Oscillating Periodic Solutions for Delay Differential Equations with Non-negativity Constraints, Siam j. Math. Anal. Vol. 47, No. 6. pp. 4467–4535

[7] Petrishyn, W.W. and Yu, Z. S. (1982). Periodic solutions of nonlinear second-order differential equations which are not solvable for the highest derivative. J. Math. Anal Appl. (89). 462-488.

[8] Rama, M. M. (1981). Ordinary differential equations theory and applications. United Kingdom.

[9] Rouche, N. and Mawhin, J. (1980). Ordinary Differential Equations. Stability and Periodic Solutions. Pitman, Boston.

[10] Royden H. L. and Fitzpatrick P. M. (2010). Real Analysis. 4th edition. Pearson Education Asia Limited and China Machine Press. China.

[11] Samoilenko, A. M. and Ronto, N. I. (1976). A Numerical-Analytic Methods for investigations of differential equations. Kiev. Ukraine

[12] Shlapak, Y. D. (1980). Periodic solutions of first-order ordinary differential equations unsolved with respect to the derivative. Ukr. Mat. Zh. (32). No. 5.638-644.

[13] Shlapak, Y. D. (1974). Periodic solutions of second-order nonlinear differential equations unsolved with respect to the highest derivative. Ukr. Mat. Zh. (26). No. 6. 850-854.

[14] Sobkovich, R. I. (1981). Periodic control problem for systems of differential equations of the second order: Analytic Methods in Nonlinear Mechanics [in Russian]. Institute of Mathematics. Ukrainian Academy of Sciences. Kiev. pp. 125-133.

[15] Tidke, H. L. and More, R. T. (2015). Existence and uniqueness of solution of integrodifferential equation in cone metric space. Sop transactions on applied mathematics.

[16] Turbaev, B. E. (1985). Periodic solutions of systems of differential equations unresolved with respect to the derivative. Visn. Kyiv. Univ. Ser. Mat. Mekh. Issue 27. 98-104.

[17] Wang, Y., Lian, H. and Ge, W. (2007). Periodic solutions for a second order nonlinear functional differential equation. Applied Mathematics Letters (20).110-115.